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ENCYCLOPAEDIA OF VEDIC MATHEMATICS

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Vedic mathematics is based on sixteen Sutras and some sub-Sutras which provide links through mathematics: the word 'Sutra' means 'thread'. Vedic scheme is not simply a collection of rapid methods; it is a system, a unified approach, which can be swiftly learnt. Vedic Mathematics extensively exploits the properties of numbers in every practical applications, particularly in the field of computation. It makes available a whole range of methods ideally suited to these properties. After a detailed introduction to Vedic Mathematics, this book shows applications of Vertically and Crosswise from basic calculations to the evaluation of logarithms, exponentials, trigonometrical functions and the solution of simultaneous, transcendental, polynomial and differential equations. The methods given in this book are not intended to be a complete or thorough treatment of the topics they deal with. All of the ideas can probably be developed further or applied in other areas, and all can doubtless be improved upon.

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गुरुकुल काँगड़ी विश्वविद्यालय, हरिद्वार

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ENCYCLOPAEDIA OF VEDIC MATHEMATICS (Vol. 2)

Shyam Priya

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Preface

Vedic Mathematics contains many examples of striking methods of calculation and there is a remarkable coherence to the system which makes it very attractive. Also the Vedic system itself suggests a kind of approach that involves going directly to the answer. Vedic mathematics is based on sixteen Sutras and some sub-Sutras which provide links through mathematics: the word 'Sutra' means 'thread'.

Vedic scheme is not simply a collection of rapid methods; it is a system, a unified approach, which can be swiftly learnt. Vedic Mathematics extensively exploits the properties of numbers in every practical applications, particularly in the field of computation. It makes available a whole range of methods ideally suited to these properties. After a detailed introduction to Vedic Mathematics, this book shows applications of Vertically and Crosswise from basic calculations to the evaluation of logarithms, exponentials, trigonometrical functions and the solution of simultaneous, transcendental, polynomial and differential equations. The methods shown are not only exceedingly efficient and new but have a unity and coherence which renders them easy to master and enjoyable to execute.

The methods given in this book are not intended to be a complete or thorough treatment of the topics they deal with. All of the ideas can probably be developed further or applied in other areas, and all can doubtless be improved upon. The mathematician will also observe a certain lack of rigor as an attempt has been made to make the material intelligible to as wide a readership as possible.

—Shyam Priya

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1

Great Hindu Mathematicians

Early Mathematicians of India

The earliest expression of mathematical understanding is inextricably linked with the origin of Hinduism. Mathematics formed an important part of the Vedas - the original Hindu scriptures - associated with ritual activities and the study of 'ganit' or mathematics was given special importance in the Vedic period.

Mathematics in Early Hindu Texts

Arithmetic operations including squares, cubes and roots are mentioned in the 'Narad Vishnu Purana' attributed to Ved Vyas (c 1000 BC), and the 1000 BC Vedic text 'Vedang Jyotish' says: "Just as the feathers of a peacock and the jewel-stone of a snake are placed at the highest point of the body, so is the position of mathematics the highest amongst all branches of the Vedas and the Shastras."

Seminal Figures in Mathematics

Hindu mathematicians, from Vedic times to the modern age have been in the forefront of making seminal contributions in the field mathematics.

Early Hindu Mathematicians

- Brahmagupta (c. 598-c. 670)
- Govindaswami (c. 800-850)

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- Mahavira (Mahaviracharya) (850)
- Pruthudakaswami (850)
- Sridhara (900)
- Manjula (930)
- Aryabhatta II (950)
- Prashastidhara (958)
- Halayudha (975)
- Jayadeva (1000)
- Sripathi (1039)
- Hemachandra Suri (b. 1089)
- Bhaskara (1114-c. 1185)
- Cangadeva (1205)
- Madhava of Sangamagramma (c. 1340-1425)
- Narayama Pandit (1350)
- Parameshvara (1360-1455)
- Nilakantha Somayaji (1455-1555)
- Shankara Variar (c. 1500-1560)
- Narayana (c. 1500-1575)
- Jyesthadeva (550)
- Acyuta Pisarati (c. 1550-1621)
- Putumana Somayaji (c. 1660-1740)
- Jaganath Pandit (1700)

Ancient Indian Mathematics

800 BC	Baudhayana
750 BC	Manava
600 BC	Apastamba
520 BC	Panini

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200 BC	Katyayana
120 AD	Yavanesvara
476	Aryabhatta I
500	Yativrsabha
505	Varahamihira
598	Brahmagupta
600	Bhaskara I
720	Lalla
800	Govindasvami
800	Mahavira
830	Prthudakasvami
840	Shankara
870	Sridhara
920	Aryabhatta II
940	Vijayanandi
1060	Brahmadeva
1114	Bhaskara II
1340	Mahendra Suri
1340	Narayana
1350	Madhava
1370	Parameshvara
1444	Nilakantha
1500	Jyesthadeva
1616	Kamalakara
1019	Sripati
1690	Jagannatha

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An Overview of Indian Mathematics

It is without doubt that mathematics today owes a huge debt to the outstanding contributions made by Indian mathematicians over many hundreds of years. What is quite surprising is that there has been a reluctance to recognise this and one has to conclude that many famous historians of mathematics found what they expected to find, or perhaps even what they hoped to find, rather than to realise what was so clear in front of them.

We shall examine the contributions of Indian mathematics in this article, but before looking at this contribution in more detail we should say clearly that the "huge debt" is the beautiful number system invented by the Indians on which much of mathematical development has rested. Laplace put this with great clarity:-

> *The ingenious method of expressing every possible number using a set of ten symbols (each symbol having a place value and an absolute value) emerged in India. The idea seems so simple nowadays that its significance and profound importance is no longer appreciated. Its simplicity lies in the way it facilitated calculation and placed arithmetic foremost amongst useful inventions the importance of this invention is more readily appreciated when one considers that it was beyond the two greatest men of Antiquity, Archimedes and Apollonius.*

We shall look briefly at the Indian development of the place-value decimal system of numbers later in this article and in somewhat more detail in the separate article Indian numerals. First, however, we go back to the first evidence of mathematics developing in India.

Histories of Indian mathematics used to begin by describing the geometry contained in the Sulbasutras but research into the history of Indian mathematics has shown

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that the essentials of this geometry were older being contained in the altar constructions described in the Vedic mythology text the Shatapatha Brahmana and the Taittiriya Samhita. Also it has been shown that the study of mathematical astronomy in India goes back to at least the third millennium BC and mathematics and geometry must have existed to support this study in these ancient times.

The first mathematics which we shall describe in this article developed in the Indus valley. The earliest known urban Indian culture was first identified in 1921 at Harappa in the Punjab and then, one year later, at Mohenjo-daro, near the Indus River in the Sindh. Both these sites are now in Pakistan but this is still covered by our term "Indian mathematics" which, in this article, refers to mathematics developed in the Indian subcontinent. The Indus civilisation (or Harappan civilisation as it is sometimes known) was based in these two cities and also in over a hundred small towns and villages. It was a civilisation which began around 2500 BC and survived until 1700 BC or later. The people were literate and used a written script containing around 500 characters which some have claimed to have deciphered but, being far from clear that this is the case, much research remains to be done before a full appreciation of the mathematical achievements of this ancient civilisation can be fully assessed.

We often think of Egyptians and Babylonians as being the height of civilisation and of mathematical skills around the period of the Indus civilisation, yet V G Childe in New Light on the Most Ancient East (1952) wrote:-

> *India confronts Egypt and Babylonia by the 3rd millennium with a thoroughly individual and independent civilisation of her own, technically the peer of the rest. And plainly it is deeply rooted in Indian soil. The Indus civilisation represents a very perfect adjustment of human life to a specific*

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environment. And it has endured; it is already specifically Indian and forms the basis of modern Indian culture.

We do know that the Harappans had adopted a uniform system of weights and measures. An analysis of the weights discovered suggests that they belong to two series both being decimal in nature with each decimal number multiplied and divided by two, giving for the main series ratios of 0.05, 0.1, 0.2, 0.5, 1, 2, 5, 10, 20, 50, 100, 200, and 500. Several scales for the measurement of length were also discovered during excavations. One was a decimal scale based on a unit of measurement of 1.32 inches (3.35 centimetres) which has been called the "Indus inch". Of course ten units is then 13.2 inches which is quite believable as the measure of a "foot". A similar measure based on the length of a foot is present in other parts of Asia and beyond. Another scale was discovered when a bronze rod was found which was marked in lengths of 0.367 inches. It is certainly surprising the accuracy with which these scales are marked. Now 100 units of this measure is 36.7 inches which is the measure of a stride. Measurements of the ruins of the buildings which have been excavated show that these units of length were accurately used by the Harappans in construction.

It is unclear exactly what caused the decline in the Harappan civilisation. Historians have suggested four possible causes: a change in climatic patterns and a consequent agricultural crisis; a climatic disaster such flooding or severe drought; disease spread by epidemic; or the invasion of Indo-Aryans peoples from the north. The favourite theory used to be the last of the four, but recent opinions favour one of the first three. What is certainly true is that eventually the Indo-Aryans peoples from the north did spread over the region. This brings us to the earliest literary record of Indian culture, the Vedas which were composed in Vedic Sanskrit, between 1500 BC and 800 BC. At first these texts, consisting of hymns, spells, and ritual

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observations, were transmitted orally. Later the texts became written works for use of those practicing the Vedic religion.

The next mathematics of importance on the Indian subcontinent was associated with these religious texts. It consisted of the Sulbasutras which were appendices to the Vedas giving rules for constructing altars. They contained quite an amount of geometrical knowledge, but the mathematics was being developed, not for its own sake, but purely for practical religious purposes. The mathematics contained in the these texts is studied in some detail in the separate article on the Sulbasutras.

The main Sulbasutras were composed by Baudhayana (about 800 BC), Manava (about 750 BC), Apastamba (about 600 BC), and Katyayana (about 200 BC). These men were both priests and scholars but they were not mathematicians in the modern sense. Although we have no information on these men other than the texts they wrote, we have included them in our biographies of mathematicians. There is another scholar, who again was not a mathematician in the usual sense, who lived around this period. That was Panini who achieved remarkable results in his studies of Sanskrit grammar. Now one might reasonably ask what Sanskrit grammar has to do with mathematics. It certainly has something to do with modern theoretical computer science, for a mathematician or computer scientist working with formal language theory will recognise just how modern some of Panini's ideas are.

Before the end of the period of the Sulbasutras, around the middle of the third century BC, the Brahmi numerals had begun to appear.

Here is one style of the Brahmi numerals..

These are the earliest numerals which, after a multitude of changes, eventually developed into the numerals 1, 2, 3, 4, 5, 6, 7, 8, 9 used today. The development of numerals and

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place-valued number systems are studied in the article Indian numerals.

The Vedic religion with its sacrificial rites began to wane and other religions began to replace it. One of these was Jainism, a religion and philosophy which was founded in India around the 6th century BC. Although the period after the decline of the Vedic religion up to the time of Aryabhatta I around 500 AD used to be considered as a dark period in Indian mathematics, recently it has been recognised as a time when many mathematical ideas were considered. In fact Aryabhatta is now thought of as summarising the mathematical developments of the Jaina as well as beginning the next phase.

The main topics of Jaina mathematics in around 150 BC were: the theory of numbers, arithmetical operations, geometry, operations with fractions, simple equations, cubic equations, quartic equations, and permutations and combinations. More surprisingly the Jaina developed a theory of the infinite containing different levels of infinity, a primitive understanding of indices, and some notion of logarithms to base 2. One of the difficult problems facing historians of mathematics is deciding on the date of the Bakhshali manuscript. If this is a work which is indeed from 400 AD, or at any rate a copy of a work which was originally written at this time, then our understanding of the achievements of Jaina mathematics will be greatly enhanced. While there is so much uncertainty over the date, a topic discussed fully in our article on the Bakhshali manuscript, then we should avoid rewriting the history of the Jaina period in the light of the mathematics contained in this remarkable document.

You can see a separate article about Jaina mathematics.

If the Vedic religion gave rise to a study of mathematics for constructing sacrificial altars, then it was Jaina cosmology which led to ideas of the infinite in Jaina mathematics.

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Later mathematical advances were often driven by the study of astronomy. Well perhaps it would be more accurate to say that astrology formed the driving force since it was that "science" which required accurate information about the planets and other heavenly bodies and so encouraged the development of mathematics. Religion too played a major role in astronomical investigations in India for accurate calendars had to be prepared to allow religious observances to occur at the correct times. Mathematics then was still an applied science in India for many centuries with mathematicians developing methods to solve practical problems.

Yavanesvara, in the second century AD, played an important role in popularising astrology when he translated a Greek astrology text dating from 120 BC. If he had made a literal translation it is doubtful whether it would have been of interest to more than a few academically minded people. He popularised the text, however, by resetting the whole work into Indian culture using Hindu images with the Indian caste system integrated into his text.

By about 500 AD the classical era of Indian mathematics began with the work of Aryabhatta. His work was both a summary of Jaina mathematics and the beginning of new era for astronomy and mathematics. His ideas of astronomy were truly remarkable. He replaced the two demons Rahu, the Dhruva Rahu which causes the phases of the Moon and the Parva Rahu which causes an eclipse by covering the Moon or Sun or their light, with a modern theory of eclipses. He introduced trigonometry in order to make his astronomical calculations, based on the Greek epicycle theory, and he solved with integer solutions indeterminate equations which arose in astronomical theories.

Aryabhatta headed a research centre for mathematics and astronomy at Kusumapura in the northeast of the Indian subcontinent. There a school studying his ideas grew up

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there but more than that, Aryabhatta set the agenda for mathematical and astronomical research in India for many centuries to come. Another mathematical and astronomical centre was at Ujjain, also in the north of the Indian subcontinent, which grew up around the same time as Kusumapura. The most important of the mathematicians at this second centre was Varahamihira who also made important contributions to astronomy and trigonometry.

The main ideas of Jaina mathematics, particularly those relating to its cosmology with its passion for large finite numbers and infinite numbers, continued to flourish with scholars such as Yativrsabha. He was a contemporary of Varahamihira and of the slightly older Aryabhatta. We should also note that the two schools at Kusumapura and Ujjain were involved in the continuing developments of the numerals and of place-valued number systems. The next figure of major importance at the Ujjain school was Brahmagupta near the beginning of the seventh century AD and he would make one of the most major contributions to the development of the numbers systems with his remarkable contributions on negative numbers and zero. It is a sobering thought that eight hundred years later European mathematics would be struggling to cope without the use of negative numbers and of zero.

These were certainly not Brahmagupta's only contributions to mathematics. Far from it for he made other major contributions in to the understanding of integer solutions to indeterminate equations and to interpolation formulas invented to aid the computation of sine tables.

The way that the contributions of these mathematicians were prompted by a study of methods in spherical astronomy is described in :-

The Hindu astronomers did not possess a general method for solving problems in spherical astronomy, unlike the Greeks who systematically followed the

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method of Ptolemy, based on the well-known theorem of Menelaus. But, by means of suitable constructions within the armillary sphere, they were able to reduce many of their problems to comparison of similar right-angled plane triangles. In addition to this device, they sometimes also used the theory of quadratic equations, or applied the method of successive approximations. ... Of the methods taught by Aryabhatta and demonstrated by his scholiast Bhaskara I, some are based on comparison of similar right-angled plane triangles, and others are derived from inference. Brahmagupta is probably the earliest astronomer to have employed the theory of quadratic equations and the method of successive approximations to solving problems in spherical astronomy.

Before continuing to describe the developments through the classical period we should explain the mechanisms which allowed mathematics to flourish in India during these centuries. The educational system in India at this time did not allow talented people with ability to receive training in mathematics or astronomy. Rather the whole educational system was family based. There were a number of families who carried the traditions of astrology, astronomy and mathematics forward by educating each new generation of the family in the skills which had been developed. We should also note that astronomy and mathematics developed on their own, separate for the development of other areas of knowledge.

Now a "mathematical family" would have a library which contained the writing of the previous generations. These writings would most likely be commentaries on earlier works such as the Aryabhatiya of Aryabhatta. Many of the commentaries would be commentaries on commentaries on commentaries etc. Mathematicians often wrote commentaries on their own work. They would not be aiming to provide texts to be used in educating people outside the family, nor

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would they be looking for innovative ideas in astronomy. Again religion was the key, for astronomy was considered to be of divine origin and each family would remain faithful to the revelations of the subject as presented by their gods. To seek fundamental changes would be unthinkable for in asking others to accept such changes would be essentially asking them to change religious belief. Nor do these men appear to have made astronomical observations in any systematic way. Some of the texts do claim that the computed data presented in them is in better agreement with observation than that of their predecessors but, despite this, there does not seem to have been a major observational programme set up. Parameshvara in the late fourteenth century appears to be one of the first Indian mathematicians to make systematic observations over many years.

Mathematics however was in a different position. It was only a tool used for making astronomical calculations. If one could produce innovative mathematical ideas then one could exhibit the truths of astronomy more easily. The mathematics therefore had to lead to the same answers as had been reached before but it was certainly good if it could achieve these more easily or with greater clarity. This meant that despite mathematics only being used as a computational tool for astronomy, the brilliant Indian scholars were encouraged by their culture to put their genius into advances in this topic.

A contemporary of Brahmagupta who headed the research centre at Ujjain was Bhaskara I who led the Asmaka school. This school would have the study of the works of Aryabhatta as their main concern and certainly Bhaskara was commentator on the mathematics of Aryabhatta. More than 100 years after Bhaskara lived the astronomer Lalla, another commentator on Aryabhatta.

The ninth century saw mathematical progress with scholars such as Govindasvami, Mahavira, Prthudakasvami,

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Shankara, and Sridhara. Some of these such as Govindasvami and Shankara were commentators on the text of Bhaskara I while Mahavira was famed for his updating of Brahmagupta's book. This period saw developments in sine tables, solving equations, algebraic notation, quadratics, indeterminate equations, and improvements to the number systems. The agenda was still basically that set by Aryabhatta and the topics being developed those in his work.

The main mathematicians of the tenth century in India were Aryabhatta II and Vijayanandi, both adding to the understanding of sine tables and trigonometry to support their astronomical calculations. In the eleventh century Sripati and Brahmadeva were major figures but perhaps the most outstanding of all was Bhaskara II in the twelfth century. He worked on algebra, number systems, and astronomy. He wrote beautiful texts illustrated with mathematical problems, some of which we present in his biography, and he provided the best summary of the mathematics and astronomy of the classical period.

Bhaskara II may be considered the high point of Indian mathematics but at one time this was all that was known:-

> *For a long time Western scholars thought that Indians had not done any original work till the time of Bhaskara II. This is far from the truth. Nor has the growth of Indian mathematics stopped with Bhaskara II. Quite a few results of Indian mathematicians have been rediscovered by Europeans. For instance, the development of number theory, the theory of indeterminates infinite series expressions for sine, cosine and tangent, computational mathematics, etc.*

Following Bhaskara II there was over 200 years before any other major contributions to mathematics were made on the Indian subcontinent. In fact for a long time it was thought that Bhaskara II represented the end of mathematical developments in the Indian subcontinent until

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modern times. However in the second half of the fourteenth century Mahendra Suri wrote the first Indian treatise on the astrolabe and Narayana wrote an important commentary on Bhaskara II, making important contributions to algebra and magic squares.

The most remarkable contribution from this period, however, was by Madhava who invented Taylor series and rigorous mathematical analysis in some inspired contributions. Madhava was from Kerala and his work there inspired a school of followers such as Nilakantha and Jyesthadeva.

Some of the remarkable discoveries of the Kerala mathematicians are described. These include: a formula for the ecliptic; the Newton-Gauss interpolation formula; the formula for the sum of an infinite series; Lhuilier's formula for the circumradius of a cyclic quadrilateral. Of particular interest is the approximation to the value of π which was the first to be made using a series. Madhava's result which gave a series for π, translated into the language of modern mathematics, reads

$$\pi R = 4R - 4R/3 + 4R/5 - \ldots$$

This formula, as well as several others referred to above, were rediscovered by European mathematicians several centuries later. Madhava also gave other formulae for π, one of which leads to the approximation 3.14159265359.

The first person in modern times to realise that the mathematicians of Kerala had anticipated some of the results of the Europeans on the calculus by nearly 300 years was Charles Whish in 1835. Whish's publication in the Transactions of the Royal Asiatic Society of Great Britain and Ireland was essentially unnoticed by historians of mathematics. Only 100 years later in the 1940s did historians of mathematics look in detail at the works of Kerala's mathematicians and find that the remarkable claims made

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by Whish were essentially true. Indeed the Kerala mathematicians had, as Whish wrote:-

> *... laid the foundation for a complete system of fluxions...*

and these works:-

> *... abound with fluxional forms and series to be found in no work of foreign countries.*

There were other major advances in Kerala at around this time. Citrabhanu was a sixteenth century mathematicians from Kerala who gave integer solutions to twenty-one types of systems of two algebraic equations. These types are all the possible pairs of equations of the following seven forms:

$x + y = a$,

$x - y = b$,

$xy = c$,

$x^2 + y^2 = d$,

$x^2 - y^2 = e$,

$x^3 + y^3 = f$, and

$x^3 - y^3 = g$.

For each case, Citrabhanu gave an explanation and justification of his rule as well as an example. Some of his explanations are algebraic, while others are geometric.

Now we have presented the latter part of the history of Indian mathematics in an unlikely way. That there would be essentially no progress between the contributions of Bhaskara II and the innovations of Madhava, who was far more innovative than any other Indian mathematician producing a totally new perspective on mathematics, seems unlikely. Much more likely is that we are unaware of the contributions made over this 200 year period which must

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have provided the foundations on which Madhava built his theories.

Our understanding of the contributions of Indian mathematicians has changed markedly over the last few decades. Much more work needs to be done to further our understanding of the contributions of mathematicians whose work has sadly been lost, or perhaps even worse, been ignored. Indeed work is now being undertaken and we should soon have a better understanding of this important part of the history of mathematics.

Baudhayana

Born: about 800 BC in India

Died: about 800 BC in India

To write a biography of Baudhayana is essentially impossible since nothing is known of him except that he was the author of one of the earliest Sulbasutras. We do not know his dates accurately enough to even guess at a life span for him, which is why we have given the same approximate birth year as death year.

He was neither a mathematician in the sense that we would understand it today, nor a scribe who simply copied manuscripts like Ahmes. He would certainly have been a man of very considerable learning but probably not interested in mathematics for its own sake, merely interested in using it for religious purposes. Undoubtedly he wrote the Sulbasutra to provide rules for religious rites and it would appear an almost certainty that Baudhayana himself would be a Vedic priest.

The mathematics given in the Sulbasutras is there to enable the accurate construction of altars needed for sacrifices. It is clear from the writing that Baudhayana, as well as being a priest, must have been a skilled craftsman. He must have been himself skilled in the practical use of

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the mathematics he described as a craftsman who himself constructed sacrificial altars of the highest quality.

The Sulbasutras are discussed in detail in the article Indian Sulbasutras. Below we give one or two details of Baudhayana's Sulbasutra, which contained three chapters, which is the oldest which we possess and, it would be fair to say, one of the two most important.

The Sulbasutra of Baudhayana contains geometric solutions (but not algebraic ones) of a linear equation in a single unknown. Quadratic equations of the forms $ax^2 = c$ and $ax^2 + bx = c$ appear.

Several values of p occur in Baudhayana's Sulbasutra since when giving different constructions Baudhayana uses different approximations for constructing circular shapes. Constructions are given which are equivalent to taking p equal to $\frac{676}{225}$ (where $\frac{676}{225} = 3.004$), $\frac{900}{289}$ (where $\frac{900}{289} = 3.114$) and to $\frac{1156}{361}$ (where $\frac{1156}{361} = 3.202$). None of these is particularly accurate but, in the context of constructing altars they would not lead to noticeable errors.

An interesting, and quite accurate, approximate value for O2 is given in Chapter 1 verse 61 of Baudhayana's Sulbasutra. The Sanskrit text gives in words what we would write in symbols as

$$\sqrt{2} = 1 + \frac{1}{3} + \frac{1}{(3+4)} - \frac{1}{(3\times4\times34)} = 577/408$$

which is, to nine places, 1.414215686. This gives O2 correct to five decimal places. This is surprising since, as we mentioned above, great mathematical accuracy did not seem necessary for the building work described. If the approximation was given as:

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$$\sqrt{2} = 1 + \frac{1}{3} + \frac{1}{(3\times4)}$$

then the error is of the order of 0.002 which is still more accurate than any of the values of π. Why then did Baudhayana feel that he had to go for a better approximation?

The Indian Sulbasutras

The Vedic people entered India about 1500 BC from the region that today is Iran. The word Vedic describes the religion of these people and the name comes from their collections of sacred texts known as the Vedas. The texts date from about the 15th to the 5th century BC and were used for sacrificial rites which were the main feature of the religion. There was a ritual which took place at an altar where food, also sometimes animals, were sacrificed. The Vedas contain recitations and chants to be used at these ceremonies. Later prose was added called Brahmanas which explained how the texts were to be used in the ceremonies. They also tell of the origin and the importance of the sacrificial rites themselves.

The Sulbasutras are appendices to the Vedas which give rules for constructing altars. If the ritual sacrifice was to be successful then the altar had to conform to very precise measurements. The people made sacrifices to their gods so that the gods might be pleased and give the people plenty food, good fortune, good health, long life, and lots of other material benefits. For the gods to be pleased everything had to be carried out with a very precise formula, so mathematical accuracy was seen to be of the utmost importance. We should also note that there were two types of sacrificial rites, one being a large public gathering while the other was a small family affair. Different types of altars were necessary for the two different types of ceremony.

All that is known of Vedic mathematics is contained in the Sulbasutras. This in itself gives us a problem, for we do

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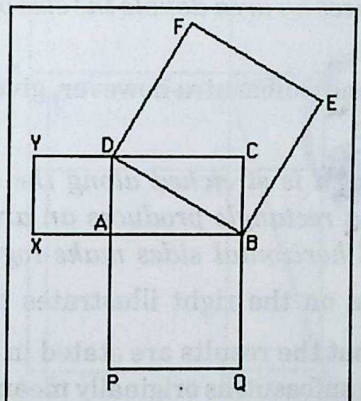
not know if these people undertook mathematical investigations for their own sake, as for example the ancient Greeks did, or whether they only studied mathematics to solve problems necessary for their religious rites. Some historians have argued that mathematics, in particular geometry, must have also existed to support astronomical work being undertaken around the same period.

Certainly the Sulbasutras do not contain any proofs of the rules which they describe. Some of the rules, such as the method of constructing a square of area equal to a given rectangle, are exact. Others, such as constructing a square of area equal to that of a given circle, are approximations. We shall look at both of these examples below but the point we wish to make here is that the Sulbasutras make no distinction between the two. Did the writers of the Sulbasutras know which methods were exact and which were approximations?



The Katyayana form of Pythagoras's theorem

The Sulbasutras were written by a scribe, although he was not the type of scribe who merely makes a copy of an existing document but one who put in considerable content and all the mathematical results may have been due to these scribes. We know nothing of the men who wrote the

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Sulbasutras other than their names and a rough indication of the period in which they lived. Like many ancient mathematicians our only knowledge of them is their writings. The most important of these documents are the Baudhayana Sulbasutra written about 800 BC and the Apastamba Sulbasutra written about 600 BC. Historians of mathematics have also studied and written about other Sulbasutras of lesser importance such as the Manava Sulbasutra written about 750 BC and the Katyayana Sulbasutra written about 200 BC.

Let us now examine some of the mathematics contained within the Sulbasutras. The first result which was clearly known to the authors is Pythagoras's theorem. The Baudhayana Sulbasutra gives only a special case of the theorem explicitly:-

The rope which is stretched across the diagonal of a square produces an area double the size of the original square.

The Katyayana Sulbasutra however, gives a more general version:-

The rope which is stretched along the length of the diagonal of a rectangle produces an area which the vertical and horizontal sides make together.

The diagram on the right illustrates this result.

Note here that the results are stated in terms of "ropes". In fact, although sulbasutras originally meant rules governing religious rites, sutras came to mean a rope for measuring an altar. While thinking of explicit statements of Pythagoras's theorem, we should note that as it is used frequently there are many examples of Pythagorean triples in the Sulbasutras. For example (5, 12, 13), (12, 16, 20), (8, 15, 17), (15, 20, 25), (12, 35, 37), (15, 36, 39), $\left(\frac{5}{2}, 6, \frac{13}{2}\right)$, and $\left(\frac{15}{2}, 10, \frac{25}{2}\right)$ all occur.

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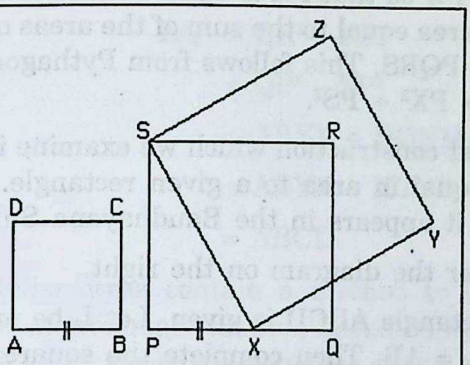
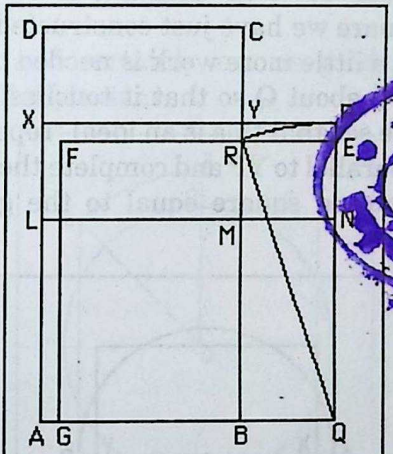
Now the Sulbasutras are really construction manuals for geometric shapes such as squares, circles, rectangles, etc. and we illustrate this with some examples.



To find a square equal to the sum of two given squares - given in all Sulbasutras



Finding a square equal to a given rectangle

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The first construction we examine occurs in most of the different Sulbasutras. It is a construction, based on Pythagoras's theorem, for making a square equal in area to two given unequal squares.

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Consider the diagram on the right.

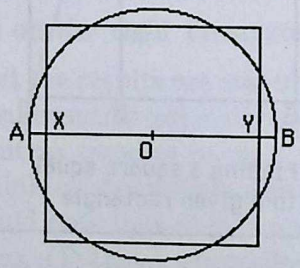
ABCD and PQRS are the two given squares. Mark a point X on PQ so that PX is equal to AB. Then the square on SX has area equal to the sum of the areas of the squares ABCD and PQRS. This follows from Pythagoras's theorem since $SX^2 = PX^2 + PS^2$.

The next construction which we examine is that to find a square equal in area to a given rectangle. We give the version as it appears in the Baudhayana Sulbasutra.

Consider the diagram on the right.

The rectangle ABCD is given. Let L be marked on AD so that AL = AB. Then complete the square ABML. Now bisect LD at X and divide the rectangle LMCD into two equal rectangles with the line XY. Now move the rectangle XYCD to the position MBQN. Complete the square AQPX.

Now the square we have just constructed is not the one we require and a little more work is needed to complete the work. Rotate PQ about Q so that it touches BY at R. Then QP = QR and we see that this is an ideal "rope" construction. Now draw RE parallel to YP and complete the square QEFG. This is the required square equal to the given rectangle ABCD.



The Sulbasutras 13/15 method of "circling the square"

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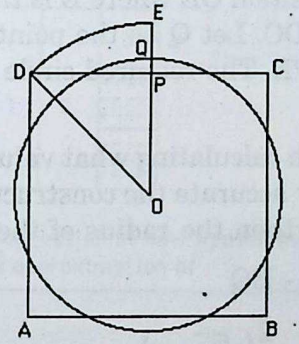
The Baudhayana Sulbasutra offers no proof of this result (or any other for that matter) but we can see that it is true by using Pythagoras's theorem.

$$\begin{aligned} EQ^2 &= QR^2 - RE^2 \\ &= QP^2 - YP^2 \\ &= ABYX + BQNM \\ &= ABYX + XYCD \\ &= ABCD. \end{aligned}$$

All the Sulbasutras contain a method to square the circle. It is an approximate method based on constructing a square of side $^{13}/_{15}$ times the diameter of the given circle as in the diagram on the right. This corresponds to taking

$\pi = \left(\frac{13}{15}\right)^2 = \frac{676}{225} = 3.00444$ so it is not a very good approximation and certainly not as good as was known earlier to the Babylonians.



The Sulbasutras method of "circling the square"

It is worth noting that many different values of p appear in the Sulbasutras, even several different ones in the same text. This is not surprising for whenever an approximate construction is given some value of p is implied. The authors

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thought in terms of approximate constructions, not in terms of exact constructions with p but only having an approximate value for it. For example in the Baudhayana Sulbasutra, as well as the value of $\frac{676}{225}$, there appears $\frac{900}{289}$ and $\frac{1156}{361}$. In different Sulbasutras the values 2.99, 3.00, 3.004, 3.029, 3.047, 3.088, 3.1141, 3.16049 and 3.2022 can all be found; see. In the value $\pi = \frac{25}{8} = 3.125$ is found in the Manava Sulbasutras.

In addition to examining the problem of squaring the circle as given by Apastamba, the authors examine the problem of dividing a segment into seven equal parts which occurs in the same Sulbasutra.

Consider the diagram on the right.

The Sulbasutras also examine the converse problem of finding a circle equal in area to a given square. The following construction appears. Given a square ABCD find the centre O. Rotate OD to position OE where E is the midpoint of the side of the square DC. Let Q be the point on PE such that PQ is one third of PE. The required circle has centre O and radius OQ.

Again it is worth calculating what value of p this implies to get a feel for how accurate the construction is. Now if the square has side 2a then the radius of the circle is r where

$$\begin{aligned} r &= OE - EQ \\ &= \sqrt{2a} - \frac{2}{3}\left(\sqrt{2a} - a\right) \\ &= a\left(\frac{\sqrt{2}}{3} + \frac{2}{3}\right). \end{aligned}$$

Then $4a^2 = \pi a^2 \left(\frac{\sqrt{2}}{3} + \frac{2}{3}\right)^2$

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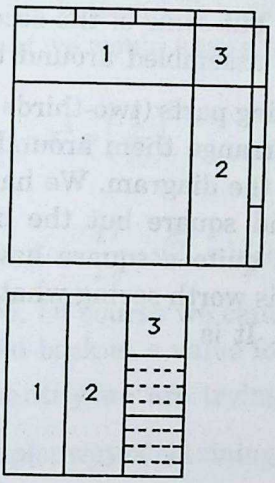
which gives $\pi = \dfrac{36}{\left(\sqrt{2}+2\right)^2} = 3.088.$

As a final look at the mathematics of the Sulbasutras we examine what may be the most remarkable. Both the Apastamba Sulbasutra and the Katyayana Sulbasutra give the following approximation to $\sqrt{2}$:

Increase a unit length by its third and this third by its own fourth less the thirty-fourth part of that fourth.



Datta's suggestion of how Apastamba found his approximation of $\sqrt{2}$

Now this gives

$$\sqrt{2} = 1 + \frac{1}{3} + \frac{1}{(3\times 4)} - \frac{1}{(3\times 4\times 34)} = \frac{577}{408}$$

which is, to nine places, 1.414215686. Compare the correct value $\sqrt{2}$ = 1.414213562 to see that the Apastamba Sulbasutra has the answer correct to five decimal places. Of course no indication is given as to how the authors of the

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Sulbasutras achieved this remarkable result. Datta, in 1932, made a beautiful suggestion as to how this approximation may have been reached.

The most likely reason for the construction was to build an altar twice the size of one already built. Datta's suggestion involves taking two squares and cutting up the second square and assembling it around the first square to give a square twice the size, thus having side $\sqrt{2}$. The second square is cut into three equal strips, and strips 1 and 2 placed around the first square as indicated in the diagram. The third strip has a square cut off the top and placed in position 3. We now have a new square but some of the second square remains and still has to be assembled around the first.

Cut the remaining parts (two-thirds of a strip) into eight equal strips and arrange them around the square we are constructing as in the diagram. We have now used all the parts of the second square but the new figure we have constructed is not quite a square having a small square corner missing. It is worth seeing what the side of this "not quite a square" is. It is

$$1+\frac{1}{3}+\frac{1}{(3\times 4)}$$

which, of course, is the first three terms of the approximation. Now Datta argues in that to improve the "not quite a square" the Sulbasutra authors could have calculated how broad a strip one needs to cut off the left hand side and bottom to fill in the missing part which has area $\left(\frac{1}{12}\right)^2$. If x is the width one cuts off then

$$2\times x\times\left(1+\frac{1}{3}+\frac{1}{12}\right)=\left(\frac{1}{12}\right)^2.$$

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This has the solution $x = \frac{1}{(3 \times 4 \times 34)}$ which is approximately 0.002450980392. We now have a square the length of whose sides is

$$1 + \frac{1}{3} + \frac{1}{(3 \times 4)} - \frac{1}{(3 \times 4 \times 34)}$$

which is exactly the approximation given by the Apastamba Sulbasutra.

Of course we have still made an approximation since the two strips of breadth x which we cut off overlapped by a square of side x in the bottom left hand corner. If we had taken this into account we would have obtained the equation

$$2 \times x \times \left(1 + \frac{1}{3} + \frac{1}{12}\right) - x^2 = \left(\frac{1}{12}\right)^2$$

for x which leads to $x = \frac{17}{12} - \sqrt{2}$ which is approximately equal to 0.002453105. Of course we cannot take this route since we have arrived back at a value for x which involves $\sqrt{2}$ which is the quantity we are trying to approximate!

Gupta gives a simpler way of obtaining the approximation for $\sqrt{2}$ than that given by Datta. He uses linear interpolation to obtain the first two terms, he then corrects the two terms so obtaining the third term, then correcting the three terms obtaining the fourth term. Although the method given by Gupta is simpler (and an interesting contribution) there is certainly something appealing in Datta's argument and somehow a feeling that this is in the spirit of the Sulbasutras.

Of course the method used by these mathematicians is very important to understanding the depth of mathematics being produced in India in the middle of the first

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millennium BC. If we follow the suggestion of some historians that the writers of the Sulbasutras were merely copying an approximation already known to the Babylonians then we might come to the conclusion that Indian mathematics of this period was far less advanced than if we follow Datta's suggestion.

Manava

Born: about 750 BC in India

Died: about 750 BC in India

Manava was the author of one of the Sulbasutras. The Manava Sulbasutra is not the oldest (the one by Baudhayana is older) nor is it one of the most important, there being at least three Sulbasutras which are considered more important. We do not know Manava's dates accurately enough to even guess at a life span for him, which is why we have given the same approximate birth year as death year. Historians disagree on 750 BC, and some would put this Sulbasutra later by one hundred or more years.

Manava would have not have been a mathematician in the sense that we would understand it today. Nor was he a scribe who simply copied manuscripts like Ahmes. He would certainly have been a man of very considerable learning but probably not interested in mathematics for its own sake, merely interested in using it for religious purposes. Undoubtedly he wrote the Sulbasutra to provide rules for religious rites and it would appear an almost certainty that Manava himself would be a Vedic priest.

The mathematics given in the Sulbasutras is there to enable accurate construction of altars needed for sacrifices. It is clear from the writing that Manava, as well as being a priest, must have been a skilled craftsman.

Manava's Sulbasutra, like all the Sulbasutras, contained approximate constructions of circles from rectangles, and

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squares from circles, which can be thought of as giving approximate values of p. There appear therefore different values of p throughout the Sulbasutra, essentially every construction involving circles leads to a different such approximation. The paper is concerned with an interpretation of verses 11.14 and 11.15 of Manava's work which give

$$\pi = \frac{25}{8} = 3.125.$$

Apastamba

Born: about 600 BC in India

Died: about 600 BC in India

To write a biography of Apastamba is essentially impossible since nothing is known of him except that he was the author of a Sulbasutra which is certainly later than the Sulbasutra of Baudhayana. It would also be fair to say that Apastamba's Sulbasutra is the most interesting from a mathematical point of view. We do not know Apastamba's dates accurately enough to even guess at a life span for him, which is why we have given the same approximate birth year as death year.

Apastamba was neither a mathematician in the sense that we would understand it today, nor a scribe who simply copied manuscripts like Ahmes. He would certainly have been a man of very considerable learning but probably not interested in mathematics for its own sake, merely interested in using it for religious purposes. Undoubtedly he wrote the Sulbasutra to provide rules for religious rites and to improve and expand on the rules which had been given by his predecessors. Apastamba would have been a Vedic priest instructing the people in the ways of conducting the religious rites he describes.

The mathematics given in the Sulbasutras is there to enable the accurate construction of altars needed for

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sacrifices. It is clear from the writing that Apastamba, as well as being a priest and a teacher of religious practices, would have been a skilled craftsman. He must have been himself skilled in the practical use of the mathematics he described as a craftsman who himself constructed sacrificial altars of the highest quality.

The Sulbasutras are discussed in detail in the article Indian Sulbasutras. Below we give one or two details of Apastamba's Sulbasutra. This work is an expanded version of that of Baudhayana. Apastamba's work consisted of six chapters while the earlier work by Baudhayana contained only three.

The general linear equation was solved in the Apastamba's Sulbasutra. He also gives a remarkably accurate value for $\sqrt{2}$ namely

$$1+\frac{1}{3}+\frac{1}{(3\times 4)}-\frac{1}{(3\times 4\times 34)}.$$

which gives an answer correct to five decimal places. A possible way that Apastamba might have reached this remarkable result is described in the article Indian Sulbasutras.

As well as the problem of squaring the circle, Apastamba considers the problem of dividing a segment into 7 equal parts. The article looks in detail at a reconstruction of Apastamba's version of these two problems.

Panini

Born: about 520 BC in Shalatula (near Attock), now Pakistan

Died: about 460 BC in India

Panini was born in Shalatula, a town near to Attock on the Indus river in present day Pakistan. The dates given for

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Panini are pure guesses. Experts give dates in the 4th, 5th, 6th and 7th century BC and there is also no agreement among historians about the extent of the work which he undertook. What is in little doubt is that, given the period in which he worked, he is one of the most innovative people in the whole development of knowledge. We will say a little more below about how historians have gone about trying to pinpoint the date when Panini lived.

Panini was a Sanskrit grammarian who gave a comprehensive and scientific theory of phonetics, phonology, and morphology. Sanskrit was the classical literary language of the Indian Hindus and Panini is considered the founder of the language and literature. It is interesting to note that the word "Sanskrit" means "complete" or "perfect" and it was thought of as the divine language, or language of the gods.

A treatise called Astadhyayi (or Astaka) is Pańini's major work. It consists of eight chapters, each subdivided into quarter chapters. In this work Panini distinguishes between the language of sacred texts and the usual language of communication. Panini gives formal production rules and definitions to describe Sanskrit grammar. Starting with about 1700 basic elements like nouns, verbs, vowels, consonants he put them into classes. The construction of sentences, compound nouns etc. is explained as ordered rules operating on underlying structures in a manner similar to modern theory. In many ways Panini's constructions are similar to the way that a mathematical function is defined today. Joseph writes:-

> *[Sanskrit's] potential for scientific use was greatly enhanced as a result of the thorough systemisation of its grammar by Panini. ... On the basis of just under 4000 sutras [rules expressed as aphorisms], he built virtually the whole structure of the Sanskrit language, whose general 'shape' hardly changed for the next two thousand years. ... An indirect consequence*

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of Panini's efforts to increase the linguistic facility of Sanskrit soon became apparent in the character of scientific and mathematical literature. This may be brought out by comparing the grammar of Sanskrit with the geometry of Euclid - a particularly apposite comparison since, whereas mathematics grew out of philosophy in ancient Greece, it was ... partly an outcome of linguistic developments in India.

Joseph goes on to make a convincing argument for the algebraic nature of Indian mathematics arising as a consequence of the structure of the Sanskrit language. In particular he suggests that algebraic reasoning, the Indian way of representing numbers by words, and ultimately the development of modern number systems in India, are linked through the structure of language.

Panini should be thought of as the forerunner of the modern formal language theory used to specify computer languages. The Backus Normal Form was discovered independently by John Backus in 1959, but Panini's notation is equivalent in its power to that of Backus and has many similar properties. It is remarkable to think that concepts which are fundamental to today's theoretical computer science should have their origin with an Indian genius around 2500 years ago.

At the beginning of this article we mentioned that certain concepts had been attributed to Panini by certain historians which others dispute. One such theory was put forward by B. Indraji in 1876. He claimed that the Brahmi numerals developed out of using letters or syllables as numerals. Then he put the finishing touches to the theory by suggesting that Panini in the eighth century BC (earlier than most historians place Panini) was the first to come up with the idea of using letters of the alphabet to represent numbers.

There are a number of pieces of evidence to support Indraji's theory that the Brahmi numerals developed from

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letters or syllables. However it is not totally convincing since, to quote one example, the symbols for 1, 2 and 3 clearly do not come from letters but from one, two and three lines respectively. Even if one accepts the link between the numerals and the letters, making Panini the originator of this idea would seem to have no more behind it than knowing that Panini was one of the most innovative geniuses that world has known so it is not unreasonable to believe that he might have made this step too.

There are other works which are closely associated with the Astadhyayi which some historians attribute to Panini, others attribute to authors before Panini, others attribute to authors after Panini. This is an area where there are many theories but few, if any, hard facts.

We also promised to return to a discussion of Panini's dates. There has been no lack of work on this topic so the fact that there are theories which span several hundreds of years is not the result of lack of effort, rather an indication of the difficulty of the topic. The usual way to date such texts would be to examine which authors are referred to and which authors refer to the work. One can use this technique and see who Panini mentions.

There are ten scholars mentioned by Panini and we must assume from the context that these ten have all contributed to the study of Sanskrit grammar. This in itself, of course, indicates that Panini was not a solitary genius but, like Newton, had "stood on the shoulders of giants". Panini must have lived later than these ten but this is absolutely no help in providing dates since we have absolutely no knowledge of when any of these ten lived.

What other internal evidence is there to use? Well of course Panini uses many phrases to illustrate his grammar any these have been examined meticulously to see if anything is contained there to indicate a date. To give an example of what we mean: if we were to pick up a text which contained

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as an example "I take the train to work every day" we would know that it had to have been written after railways became common. Let us illustrate with two actual examples from the Astadhyayi which have been the subject of much study. The first is an attempt to see whether there is evidence of Greek influence. Would it be possible to find evidence which would mean that the text had to have been written after the conquests of Alexander the Great? There is a little evidence of Greek influence, but there was Greek influence on this north east part of the Indian subcontinent before the time of Alexander. Nothing conclusive has been identified.

Another angle is to examine a reference Panini makes to nuns. Some argue that these must be Buddhist nuns and therefore the work must have been written after Buddha. A nice argument but there is a counter argument which says that there were Jaina nuns before the time of Buddha and Panini's reference could equally well be to them. Again the evidence is inconclusive.

There are references by others to Panini. However it would appear that the Panini to whom most refer is a poet and although some argue that these are the same person, most historians agree that the linguist and the poet are two different people. Again this is inconclusive evidence. Let us end with an evaluation of Panini's contribution by Cardona:-

> *Panini's grammar has been evaluated from various points of view. After all these different evaluations, I think that the grammar merits asserting ... that it is one of the greatest monuments of human intelligence.*

Katyayana

Born: about 200 BC in India

Died: about 200 BC in India

We cannot attempt to write a biography of Katyayana since essentially nothing is known of him except that he was

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the author of a Sulbasutra which is much later than the Sulbasutras of Baudhayana and Apastamba. It would also be fair to say that Katyayana's Sulbasutra is the least interesting from a mathematical point of view of the three best known Sulbasutras. It adds very little to that of Apastamba written several hundreds of years earlier. We do not know Katyayana's dates accurately enough to even guess at a life span for him, which is why we have given the same approximate birth year as death year.

Katyayana was neither a mathematician in the sense that we would understand it today, nor a scribe who simply copied manuscripts like Ahmes. He would certainly have been a man of very considerable learning but probably not interested in mathematics for its own sake, merely interested in using it for religious purposes. Undoubtedly he wrote the Sulbasutra to provide rules for religious rites and to improve and expand on the rules which had been given by his predecessors. Katyayana would have been a priest instructing the people in the ways of conducting the religious rites he describes.

Katyayana lived in a period when the religious rites that the Sulbasutras were written to support were becoming less influential. People were turning to other religions and perhaps this lack of vigour in the religion at this time partly explains why several hundreds of years after Apastamba Katyayana adds little of importance to the Sulbasutra which he wrote.

Yavanesvara

Born: about 120 in Western India

Died: about 180 in India

Indian astrology was originally known as Jyotisha, which means "science of the stars". Until around the first century AD no real distinction was made between astrology and astronomy and in fact most astronomical theories were

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propounded to support the theory that the positions of the heavenly bodies directly influenced human events.

The Indian methods of computing horoscopes all date back to the translation of a Greek astrology text into Sanskrit prose by Yavanesvara in 149 AD. Yavanesvara (or Yavanaraja) literally means "Lord of the Greeks" and it was a name given to many officials in western India during the period 130 AD - 390 AD. During this period the Ksatrapas ruled Gujarat (or Madhya Pradesh) and these "Lord of the Greeks" officials acted for the Greek merchants living in the area.

The particular "Lord of the Greeks" official Yavanesvara who we are interested in here worked under Rudradaman. Rudradaman became ruler of the Ksatrapas in around 130 AD and it was during the period of his rule that Yavanesvara worked as an official and made his translation. We know of Rudradaman because information is recorded in a lengthy Sanskrit inscription at Junagadh written around 150 AD.

The Greek astrology text in question was written in Alexandria some time round about 120 BC. Yavanesvara did far more than just translate the Greek text for such a translation would have had little relevance to the Indians. He therefore not only translated the language but he translated the context too. Instead of the Greek gods who appear in the original, Yavanesvara used Hindu images. Again he worked the Indian caste system into the work and made the work one which would fit well with the Indian thought.

The work was written with the aim of letting Indians became astrologers so it had to present astronomy in a form in which it could be used for astrology. In order to do this Yavanesvara put into his work an explanation of the Greek version of the Babylonian theory of the motions of the planets. All this he wrote in Sanskrit prose but sadly the original has not survived. We do have, however, a version written

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in Sanskrit verse 120 years after Yavanesvara's work appeared.

Yavanesvara had an important influence on the whole of astrology in India for centuries after he made his popular translation. Although the influence was more than on astrology, as the science of astronomy split from astrology, the influence of Yavanesvara's work reached into astronomy too.

Aryabhatta the Elder

Born: 476 in Kusumapura (now Patna), India

Died: 550 in India

Aryabhatta is also known as Aryabhatta I to distinguish him from the later mathematician of the same name who lived about 400 years later. Al-Biruni has not helped in understanding Aryabhatta's life, for he seemed to believe that there were two different mathematicians called Aryabhatta living at the same time. He therefore created a confusion of two different Aryabhatas which was not clarified until 1926 when B Datta showed that al-Biruni's two Aryabhatas were one and the same person.

We know the year of Aryabhatta's birth since he tells us that he was twenty-three years of age when he wrote Aryabhatiya which he finished in 499. We have given Kusumapura, thought to be close to Pataliputra (which was refounded as Patna in Bihar in 1541), as the place of Aryabhatta's birth but this is far from certain, as is even the location of Kusumapura itself. As Parameswaran writes:-

> *... no final verdict can be given regarding the locations of Asmakajanapada and Kusumapura.*

We do know that Aryabhatta wrote Aryabhatiya in Kusumapura at the time when Pataliputra was the capital of the Gupta empire and a major centre of learning, but there have been numerous other places proposed by historians

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as his birthplace. Some conjecture that he was born in south India, perhaps Kerala, Tamil Nadu or Andhra Pradesh, while others conjecture that he was born in the north-east of India, perhaps in Bengal. It is claimed that Aryabhatta was born in the Asmaka region of the Vakataka dynasty in South India although the author accepted that he lived most of his life in Kusumapura in the Gupta empire of the north. However, giving Asmaka as Aryabhaṭta's birthplace rests on a comment made by Nilakantha Somayaji in the late 15th century. It is now thought by most historians that Nilakantha confused Aryabhatta with Bhaskara I who was a later commentator on the Aryabhatiya.

We should note that Kusumapura became one of the two major mathematical centres of India, the other being Ujjain. Both are in the north but Kusumapura (assuming it to be close to Pataliputra) is on the Ganges and is the more northerly. Pataliputra, being the capital of the Gupta empire at the time of Aryabhatta, was the centre of a communications network which allowed learning from other parts of the world to reach it easily, and also allowed the mathematical and astronomical advances made by Aryabhatta and his school to reach across India and also eventually into the Islamic world.

As to the texts written by Aryabhatta only one has survived. However Jha claims that:-

> *... Aryabhatta was an author of at least three astronomical texts and wrote some free stanzas as well.*

The surviving text is Aryabhatta's masterpiece the Aryabhatiya which is a small astronomical treatise written in 118 verses giving a summary of Hindu mathematics up to that time. Its mathematical section contains 33 verses giving 66 mathematical rules without proof. The Aryabhatiya contains an introduction of 10 verses, followed by a section on mathematics with, as we just mentioned, 33 verses, then

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a section of 25 verses on the reckoning of time and planetary models, with the final section of 50 verses being on the sphere and eclipses.

There is a difficulty with this layout which is discussed in detail by van der Waerden. Van der Waerden suggests that in fact the 10 verse Introduction was written later than the other three sections. One reason for believing that the two parts were not intended as a whole is that the first section has a different meter to the remaining three sections. However, the problems do not stop there. We said that the first section had ten verses and indeed Aryabhatta titles the section Set of ten giti stanzas. But it in fact contains eleven giti stanzas and two arya stanzas. Van der Waerden suggests that three verses have been added and he identifies a small number of verses in the remaining sections which he argues have also been added by a member of Aryabhatta's school at Kusumapura.

The mathematical part of the Aryabhatiya covers arithmetic, algebra, plane trigonometry and spherical trigonometry. It also contains continued fractions, quadratic equations, sums of power series and a table of sines. Let us examine some of these in a little more detail.

First we look at the system for representing numbers which Aryabhatta invented and used in the Aryabhatiya. It consists of giving numerical values to the 33 consonants of the Indian alphabet to represent 1, 2, 3, ... , 25, 30, 40, 50, 60, 70, 80, 90, 100. The higher numbers are denoted by these consonants followed by a vowel to obtain 100, 10000, ... In fact the system allows numbers up to 1018 to be represented with an alphabetical notation. Ifrah argues that Aryabhatta was also familiar with numeral symbols and the place-value system. He writes:-

> *... it is extremely likely that Aryabhatta knew the sign for zero and the numerals of the place value system. This supposition is based on the following two facts:*

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first, the invention of his alphabetical counting system would have been impossible without zero or the place-value system; secondly, he carries out calculations on square and cubic roots which are impossible if the numbers in question are not written according to the place-value system and zero.

Next we look briefly at some algebra contained in the Aryabhatiya. This work is the first we are aware of which examines integer solutions to equations of the form by =ax+c and by = ax - c, where a, b, c are integers. The problem arose from studying the problem in astronomy of determining the periods of the planets. Aryabhatta uses the kuttaka method to solve problems of this type. The word kuttaka means "to pulverise" and the method consisted of breaking the problem down into new problems where the coefficients became smaller and smaller with each step. The method here is essentially the use of the Euclidean algorithm to find the highest common factor of a and b but is also related to continued fractions.

Aryabhatta gave an accurate approximation for p. He wrote in the Aryabhatiya the following:

Add four to one hundred, multiply by eight and then add sixty-two thousand the result is approximately the circumference of a circle of diameter twenty thousand. By this rule the relation of the circumference to diameter is given.

This gives $\pi = \frac{62832}{20000}$ = 3.1416 which is a surprisingly accurate value. In fact π = 3.14159265 correct to 8 places. If obtaining a value this accurate is surprising, it is perhaps even more surprising that Aryabhatta does not use his accurate value for p but prefers to use O10 = 3.1622 in practice. Aryabhatta does not explain how he found this accurate value but, for example, Ahmad considers this value as an approximation to half the perimeter of a regular

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polygon of 256 sides inscribed in the unit circle. However, Bruins shows that this result cannot be obtained from the doubling of the number of sides. Another interesting paper discussing this accurate value of π by Aryabhatta where Jha writes:

> *Aryabhatta I's value of π is a very close approximation to the modern value and the most accurate among those of the ancients. There are reasons to believe that Aryabhatta devised a particular method for finding this value. It is shown with sufficient grounds that Aryabhatta himself used it, and several later Indian mathematicians and even the Arabs adopted it. The conjecture that Aryabhatta's value of π is of Greek origin is critically examined and is found to be without foundation. Aryabhatta discovered this value independently and also realised that π is an irrational number. He had the Indian background, no doubt, but excelled all his predecessors in evaluating π. Thus the credit of discovering this exact value of π may be ascribed to the celebrated mathematician, Aryabhatta I.*

We now look at the trigonometry contained in Aryabhatta's treatise. He gave a table of sines calculating the approximate values at intervals of 90°/24 = 3°45'. In order to do this he used a formula for sin(n+1)x - sin nx in terms of sin nx and sin (n-1)x. He also introduced the versine (versin = 1 - cosine) into trigonometry.

Other rules given by Aryabhatta include that for summing the first n integers, the squares of these integers and also their cubes. Aryabhatta gives formulae for the areas of a triangle and of a circle which are correct, but the formulae for the volumes of a sphere and of a pyramid are claimed to be wrong by most historians. For example Ganitanand describes as "mathematical lapses" the fact that Aryabhatta gives the incorrect formula V = Ah/2 for the volume of a

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pyramid with height h and triangular base of area A. He also appears to give an incorrect expression for the volume of a sphere. However, as is often the case, nothing is as straightforward as it appears and Elfering argues that this is not an error but rather the result of an incorrect translation.

This relates to verses 6, 7, and 10 of the second section of the Aryabhatiya and Elfering produces a translation which yields the correct answer for both the volume of a pyramid and for a sphere. However, in his translation Elfering translates two technical terms in a different way to the meaning which they usually have. Without some supporting evidence that these technical terms have been used with these different meanings in other places it would still appear that Aryabhatta did indeed give the incorrect formulae for these volumes.

We have looked at the mathematics contained in the Aryabhatiya but this is an astronomy text so we should say a little regarding the astronomy which it contains. Aryabhatta gives a systematic treatment of the position of the planets in space. He gave the circumference of the earth as 4 967 yojanas and its diameter as $1\ 581\frac{1}{24}$ yojanas.

Since 1 yojana = 5 miles this gives the circumference as 24 835 miles, which is an excellent approximation to the currently accepted value of 24 902 miles. He believed that the apparent rotation of the heavens was due to the axial rotation of the Earth. This is a quite remarkable view of the nature of the solar system which later commentators could not bring themselves to follow and most changed the text to save Aryabhatta from what they thought were stupid errors!

Aryabhatta gives the radius of the planetary orbits in terms of the radius of the Earth/Sun orbit as essentially their periods of rotation around the Sun. He believes that the Moon and planets shine by reflected sunlight, incredibly

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he believes that the orbits of the planets are ellipses. He correctly explains the causes of eclipses of the Sun and the Moon. The Indian belief up to that time was that eclipses were caused by a demon called Rahu. His value for the length of the year at 365 days 6 hours 12 minutes 30 seconds is an overestimate since the true value is less than 365 days 6 hours.

Bhaskara I who wrote a commentary on the Aryabhatiya about 100 years later wrote of Aryabhatta:-

> *Aryabhatta is the master who, after reaching the furthest shores and plumbing the inmost depths of the sea of ultimate knowledge of mathematics, kinematics and spherics, handed over the three sciences to the learned world.*

Yativrsabha

Born: about 500 in India

Died: about 570 in India

Yativrsabha (or Jadivasaha) was a Jaina mathematician who studied under Arya Manksu and Nagahastin. We know nothing of Yativrsabha's dates except for a reference which he makes to the end of the Gupta dynasty which he says was after 231 years of ruling. This ended in 551 so we must assume that 551 AD is a date which occurred during Yativrsabha's lifetime. This fits with the only other information regarding his dates which are that his work is referenced by Jinabhadra Ksamasramana in 609 and that Yativrsabha himself refers to a work written by Sarvanandin in 458.

Yativrsabha's work Tiloyapannatti gives various units for measuring distances and time and also describes the system of infinite time measures. It is a work which describes Jaina cosmology and gives a description of the universe which is of historical importance in understanding Jaina

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science and mathematics. The Jaina belief was in an infinite world, both infinite in space and in time. This led the Jainas to devise ways of measuring larger and larger distances and longer and longer intervals of time. It led them to consider different measures of infinity, and in this respect the Jaina mathematicians would appear to be the only ones before the time when Cantor developed the theory of infinite cardinals to envisage different magnitudes of infinity.

Varahamihira

Born: 505 in Kapitthaka, India

Died: 587 in India

Our knowledge of Varahamihira is very limited indeed. According to one of his works, he was educated in Kapitthaka. However, far from settling the question this only gives rise to discussions of possible interpretations of where this place was. Dhavale discusses this problem. We do not know whether he was born in Kapitthaka, wherever that may be, although we have given this as the most likely guess.

We do know, however, that he worked at Ujjain which had been an important centre for mathematics since around 400 AD. The school of mathematics at Ujjain was increased in importance due to Varahamihira working there and it continued for a long period to be one of the two leading mathematical centres in India, in particular having Brahmagupta as its next major figure.

The most famous work by Varahamihira is the Pancasiddhantika (The Five Astronomical Canons) dated 575 AD. This work is important in itself and also in giving us information about older Indian texts which are now lost. The work is a treatise on mathematical astronomy and it summarises five earlier astronomical treatises, namely the Surya, Romaka, Paulisa, Vasistha and Paitamaha siddhantas. Shukla states:-

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The Pancasiddhantika of Varahamihira is one of the most important sources for the history of Hindu astronomy before the time of Aryabhatta II.

One treatise which Varahamihira summarises was the Romaka-Siddhanta which itself was based on the epicycle theory of the motions of the Sun and the Moon given by the Greeks in the 1st century AD. The Romaka-Siddhanta was based on the tropical year of Hipparchus and on the Metonic cycle of 19 years. Other works which Varahamihira summarises are also based on the Greek epicycle theory of the motions of the heavenly bodies. He revised the calendar by updating these earlier works to take into account precession since they were written. The Pancasiddhantika also contains many examples of the use of a place-value number system.

There is, however, quite a debate about interpreting data from Varahamihira's astronomical texts and from other similar works. Some believe that the astronomical theories are Babylonian in origin, while others argue that the Indians refined the Babylonian models by making observations of their own. Much needs to be done in this area to clarify some of these interesting theories.

Ifrah notes that Varahamihira was one of the most famous astrologers in Indian history. His work Brihatsamhita (The Great Compilation) discusses topics such as:-

... descriptions of heavenly bodies, their movements and conjunctions, meteorological phenomena, indications of the omens these movements, conjunctions and phenomena represent, what action to take and operations to accomplish, sign to look for in humans, animals, precious stones, etc.

Varahamihira made some important mathematical discoveries. Among these are certain trigonometric formulae which translated into our present day notation correspond to:

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$\sin x = \cos(\pi/2 - x)$,

$\sin^2 x + \cos^2 x = 1$, and

$(1 - \cos 2x)/2 = \sin^2 x$.

Another important contribution to trigonometry was his sine tables where he improved those of Aryabhatta I giving more accurate values.

It should be emphasised that accuracy was very important for these Indian mathematicians since they were computing sine tables for applications to astronomy and astrology. This motivated much of the improved accuracy they achieved by developing new interpolation methods.

The Jaina school of mathematics investigated rules for computing the number of ways in which r objects can be selected from n objects over the course of many hundreds of years. They gave rules to compute the binomial coefficients nCr which amount to

$${}_nC_r = n(n-1)(n-2)...(n-r+1)/r!$$

However, Varahamihira attacked the problem of computing ${}_nC_r$ in a rather different way. He wrote the numbers n in a column with n = 1 at the bottom. He then put the numbers r in rows with r = 1 at the left-hand side. Starting at the bottom left side of the array which corresponds to the values n = 1, r = 1, the values of ${}_nC_r$ are found by summing two entries, namely the one directly below the (n, r) position and the one immediately to the left of it. Of course this table is none other than Pascal's triangle for finding the binomial coefficients despite being viewed from a different angle from the way we build it up today. Full details of this work by Varahamihira is given.

Hayashi, examines Varahamihira's work on magic squares. In particular he examines a pandiagonal magic square of order four which occurs in Varahamihira's work.

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Brahmagupta

Born: 598 in (possibly) Ujjain, India

Died: 670 in India

Brahmagupta, whose father was Jisnugupta, wrote important works on mathematics and astronomy. In particular he wrote Brahmasphutasiddhanta (The Opening of the Universe), in 628. The work was written in 25 chapters and Brahmagupta tells us in the text that he wrote it at Bhillamala which today is the city of Bhinmal. This was the capital of the lands ruled by the Gurjara dynasty.

Brahmagupta became the head of the astronomical observatory at Ujjain which was the foremost mathematical centre of ancient India at this time. Outstanding mathematicians such as Varahamihira had worked there and built up a strong school of mathematical astronomy.

In addition to the Brahmasphutasiddhanta Brahmagupta wrote a second work on mathematics and astronomy which is the Khandakhadyaka written in 665 when he was 67 years old. We look below at some of the remarkable ideas which Brahmagupta's two treatises contain. First let us give an overview of their contents.

The Brahmasphutasiddhanta contains twenty-five chapters but the first ten of these chapters seem to form what many historians believe was a first version of Brahmagupta's work and some manuscripts exist which contain only these chapters. These ten chapters are arranged in topics which are typical of Indian mathematical astronomy texts of the period.

The topics covered are: mean longitudes of the planets; true longitudes of the planets; the three problems of diurnal rotation; lunar eclipses; solar eclipses; risings and settings; the moon's crescent; the moon's shadow; conjunctions of the planets with each other; and conjunctions of the planets with the fixed stars.

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The remaining fifteen chapters seem to form a second work which is major addendum to the original treatise. The chapters are: examination of previous treatises on astronomy; on mathematics; additions to chapter 1; additions to chapter 2; additions to chapter 3; additions to chapter 4 and 5; additions to chapter 7; on algebra; on the gnomon; on meters; on the sphere; on instruments; summary of contents; versified tables.

Brahmagupta's understanding of the number systems went far beyond that of others of the period. In the Brahmasphutasiddhanta he defined zero as the result of subtracting a number from itself. He gave some properties as follows:-

When zero is added to a number or subtracted from a number, the number remains unchanged; and a number multiplied by zero becomes zero.

He also gives arithmetical rules in terms of fortunes (positive numbers) and debts (negative numbers):-

A debt minus zero is a debt.

A fortune minus zero is a fortune.

Zero minus zero is a zero.

A debt subtracted from zero is a fortune.

A fortune subtracted from zero is a debt.

The product of zero multiplied by a debt or fortune is zero.

The product of zero multipliedby zero is zero.

The product or quotient of two fortunes is one fortune.

The product or quotient of two debts is one fortune.

The product or quotient of a debt and a fortune is a debt.

The product or quotient of a fortune and a debt is a debt.

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Brahmagupta then tried to extend arithmetic to include division by zero:-

Positive or negative numbers when divided by zero is a fraction the zero as denominator.

Zero divided by negative or positive numbers is either zero or is expressed as a fraction with zero as numerator and the finite quantity as denominator.

Zero divided by zero is zero.

Really Brahmagupta is saying very little when he suggests that n divided by zero is n/0. He is certainly wrong when he then claims that zero divided by zero is zero. However it is a brilliant attempt to extend arithmetic to negative numbers and zero.

We can also describe his methods of multiplication which use the place-value system to its full advantage in almost the same way as it is used today. We give three examples of the methods he presents in the Brahmasphuta siddhanta and in doing so we follow Ifrah. The first method we describe is called "gomutrika" by Brahmagupta. Ifrah translates "gomutrika" to "like the trajectory of a cow's urine". Consider the product of 235 multiplied by 264. We begin by setting out the sum as follows:

```
2 235
6 235
4 235
----------
```

Now multiply the 235 of the top row by the 2 in the top position of the left hand column. Begin by 2 5 = 10, putting 0 below the 5 of the top row, carrying 1 in the usual way to get

```
2 235
6 235
4 235
----------
 470
```

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Now multiply the 235 of the second row by the 6 in the left hand column writing the number in the line below the 470 but moved one place to the right

```
2 235
6 235
4 235
----------
 470
 1410
```

Now multiply the 235 of the third row by the 4 in the left hand column writing the number in the line below the 1410 but moved one place to the right

```
2 235
6 235
4 235
----------
 470
 1410
 940
```

Now add the three numbers below the line

```
2 235
6 235
4 235
----------
 470
 1410
 940
----------
 62040
```

The variants are first writing the second number on the right but with the order of the digits reversed as follows

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```
235 4
235 6
235 2
----------
940
1410
470
----------
62040
```

The third variant just writes each number once but otherwise follows the second method

```
235
----------
940 4
1410 6
470 2
----------
62040
```

Another arithmetical result presented by Brahmagupta is his algorithm for computing square roots. This algorithm is discussed where it is shown to be equivalent to the Newton-Raphson iterative formula.

Brahmagupta developed some algebraic notation and presents methods to solve quardatic equations. He presents methods to solve indeterminate equations of the form ax + c = by. Majumdar writes:-

> *Brahmagupta perhaps used the method of continued fractions to find the integral solution of an indeterminate equation of the type ax + c = by.*

Majumdar gives the original Sanskrit verses from Brahmagupta's Brahmasphuta siddhanta and their English translation with modern interpretation.

Brahmagupta also solves quadratic indeterminate equations of the type $ax^2 + c = y^2$ and $ax^2 - c = y^2$. For example

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he solves $8x^2 + 1 = y^2$ obtaining the solutions (x,y) = (1,3), (6,17), (35,99), (204,577), (1189,3363), ... For the equation $11x^2 + 1 = y^2$ Brahmagupta obtained the solutions (x,y) = (3,10), (161/5,534/5), ... He also solves $61x^2 + 1 = y^2$ which is particularly elegant having x = 226153980, y = 1766319049 as its smallest solution.

A example of the type of problems Brahmagupta poses and solves in the Brahmasphutasiddhanta is the following:-

Five hundred drammas were loaned at an unknown rate of interest, The interest on the money for four months was loaned to another at the same rate of interest and amounted in ten mounths to 78 drammas. Give the rate of interest.

Rules for summing series are also given. Brahmagupta gives the sum of the squares of the first n natural numbers as n(n+1)(2n+1)/6 and the sum of the cubes of the first n natural numbers as $(n(n+1)/2)^2$. No proofs are given so we do not know how Brahmagupta discovered these formulae.

In the Brahmasphutasiddhanta Brahmagupta gave remarkable formulae for the area of a cyclic quadrilateral and for the lengths of the diagonals in terms of the sides. The only debatable point here is that Brahmagupta does not state that the formulae are only true for cyclic quadrilaterals so some historians claim it to be an error while others claim that he clearly meant the rules to apply only to cyclic quadrilaterals.

Much material in the Brahmasphutasiddhanta deals with solar and lunar eclipses, planetary conjunctions and positions of the planets. Brahmagupta believed in a static Earth and he gave the length of the year as 365 days 6 hours 5 minutes 19 seconds in the first work, changing the value to 365 days 6 hours 12 minutes 36 seconds in the second book the Khandakhadyaka. This second values is not, of course, an improvement on the first since the true length of the years if less than 365 days 6 hours. One has to wonder

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whether Brahmagupta's second value for the length of the year is taken from Aryabhatta I since the two agree to within 6 seconds, yet are about 24 minutes out.

The Khandakhadyaka is in eight chapters again covering topics such as: the longitudes of the planets; the three problems of diurnal rotation; lunar eclipses; solar eclipses; risings and settings; the moon's crescent; and conjunctions of the planets. It contains an appendix which is some versions has only one chapter, in other versions has three.

Of particular interest to mathematics in this second work by Brahmagupta is the interpolation formula he uses to compute values of sines. This is studied in detail where it is shown to be a particular case up to second order of the more general Newton-Stirling interpolation formula.

Bhaskara I

Born: about 600 in (possibly) Saurastra (modern Gujarat state), India

Died: about 680 in (possibly) Asmaka, India

We have very little information about Bhaskara I's life except what can be deduced from his writings. Shukla deduces from the fact that Bhaskara I often refers to the Asmakatantra instead of the Aryabhatiya that he must have been working in a school of mathematicians in Asmaka which was probably in the Nizamabad District of Andhra Pradesh. If this is correct, and it does seem quite likely, then the school in Asmaka would have been a collection of scholars who were followers of Aryabhatta I and of course this fits in well with the fact that Bhaskara I himself was certainly a follower of Aryabhatta I.

There are other references to places in India in Bhaskara's writings. For example he mentions Valabhi (today Vala), the capital of the Maitraka dynasty in the 7th century, and Sivarajapura, which were both in Saurastra which today is

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the Gujarat state of India on the west coast of the continent. Also mentioned are Bharuch (or Broach) in southern Gujarat and Thanesar in the eastern Punjab which was ruled by Harsa for 41 years from 606. Harsa was the preeminent ruler in north India through the first half of Bhaskara I's life. A reasonable guess would be that Bhaskara was born in Saurastra and later moved to Asmaka.

Bhaskara I was an author of two treatises and commentaries to the work of Aryabhatta I. His works are the Mahabhaskariya, the Laghubhaskariya and the Aryabhatiyabhasya. The Mahabhaskariya is an eight chapter work on Indian mathematical astronomy and includes topics which were fairly standard for such works at this time. It discusses topics such as: the longitudes of the planets; conjunctions of the planets with each other and with bright stars; eclipses of the sun and the moon; risings and settings; and the lunar crescent.

Bhaskara I included in his treatise the Mahabhaskariya three verses which give an approximation to the trigonometric sine function by means of a rational fraction. These occur in Chapter 7 of the work. The formula which Bhaskara gives is amazingly accurate and use of the formula leads to a maximum error of less than one percent. The formula is

$$\sin \xi = 16\xi\ (\pi - x)/[5\pi^2 - 4x\ (\pi - x)]$$

and Bhaskara attributes the work as that of Aryabhatta I. We have computed the values given by the formula and compared it with the correct value for sin x for x from 0 to p/2 in steps of $\pi/20$.

$x = 0$	formula = 0.00000	sin x = 0.00000	error = 0.00000
$x = \pi/20$	formula = 0.15800	sin x = 0.15643	error = 0.00157
$x = \pi/10$	formula = 0.31034	sin x = 0.30903	error = 0.00131
$x = 3\pi/20$	formula = 0.45434	sin x = 0.45399	error = 0.00035
$x = \pi/5$	formula = 0.58716	sin x = 0.58778	error = -0.00062
$x = \pi/4$	formula = 0.70588	sin x = 0.70710	error = -0.00122

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$x = \pi/10$	formula = 0.80769	sin x = 0.80903	error = -0.00134
$x = 7\pi/20$	formula = 0.88998	sin x = 0.89103	error = -0.00105
$x = 2\pi/5$	formula = 0.95050	sin x = 0.95105	error = -0.00055
$x = 9\pi/20$	formula = 0.98753	sin x = 0.98769	error = -0.00016
$x = \pi/2$	formula = 1.00000	sin x = 1.00000	error = 0.00000

In 629 Bhaskara I wrote a commentary, the Aryabhatiyabhasya, on the Aryabhatiya by Aryabhatta I. The Aryabhatiya contains 33 verses dealing with mathematics, the remainder of the work being concerned with mathematical astronomy. The commentary by Bhaskara I is only on the 33 verses of mathematics. He considers problems of indeterminate equations of the first degree and trigonometric formulae. In the course of discussions of the Aryabhatiya, Bhaskara I expressed his idea on how one particular rectangle can be treated as a cyclic quadrilateral. He was the first to open discussion on quadrilaterals with all the four sides unequal and none of the opposite sides parallel.

One of the approximations used for p for many centuries was O10. Bhaskara I criticised this approximation. He regretted that an exact measure of the circumference of a circle in terms of diameter was not available and he clearly believed that p was not rational.

Shukla discusses some features of Bhaskara's mathematics such as: numbers and symbolism, the classification of mathematics, the names and solution methods of equations of the first degree, quadratic equations, cubic equations and equations with more than one unknown, symbolic algebra, unusual and special terms in Bhaskara's work, weights and measures, the Euclidean algorithm method of solving linear indeterminate equations, examples given by Bhaskara I illustrating Aryabhatta I's rules, certain tables for solving an equation occurring in astronomy, and reference made by Bhaskara I to the works of earlier Indian mathematicians.

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Lalla

Born: about 720 in India

Died: about 790 in India

Lalla's father was Trivikrama Bhatta and Trivikrama's father, Lalla's paternal grandfather, was named Samba. Lalla was an Indian astronomer and mathematician who followed the tradition of Aryabhatta I. Lalla's most famous work was entitled Shishyadhividdhidatantra. This major treatise was in two volumes. The first volume, On the computation of the positions of the planets, was in thirteen chapters and covered topics such as: mean longitudes of the planets; true longitudes of the planets; the three problems of diurnal rotation; lunar eclipses; solar eclipses; syzygies; risings and settings; the shadow of the moon; the moon's crescent; conjunctions of the planets with each other; conjunctions of the planets with the fixed stars; the patas of the moon and sun, and a final chapter in the first volume which forms a conclusion.

The second volume was On the sphere. In this volume Lalla examined topics such as: graphical representation; the celestial sphere; the principle of mean motion; the terrestrial sphere; motions and stations of the planets; geography; erroneous knowledge; instruments; and finally selected problems.

In Shishyadhividdhidatantra Lalla uses Sanskrit numerical symbols. Ifrah writes:-

> *... over the centuries, Sanskrit has lent itself admirably to the rules of prosody and versification. This explains why Indian astronomers [like Lalla] favoured the use of Sanskrit numerical symbols, based on a complex symbolism which was extraordinarily fertile and sophisticated, possessing as it did an almost limitless choice of synonyms.*

Despite writing the most famous treatise giving the views of Aryabhatta I, Lalla did not accept his theory given

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in the Aryabhatiya that the earth rotated. Lalla argues in his commentary, like many other Indian astronomers before him such as Varahamihira and Brahmagupta, that if the earth rotated then the speed would have to be such that one would have to ask how do the bees or birds flying in the sky come back to their nests? In fact Lalla misinterpreted some of Aryabhatta I's statements about the rotating earth. One has to assume that the idea appeared so impossible to him that he just could not appreciate Aryabhatta I's arguments. As Chatterjee writes, Lalla in his commentary:-

> *... did not interpret the relevant verses in the way meant by Aryabhatta I.*

Astrology at this time was based on astronomical tables and often the horoscopes allow one to identify the tables used. Some Arabic horoscopes were based on astronomical tables calculated in India. The most frequently used tables were by Aryabhatta I. Lalla improved on these tables and he produced a set of corrections for the Moon's longitude. One aspect of Aryabhatta I's work which Lalla did follow was his value of π. Lalla uses $\pi = \frac{62832}{20000}$, i.e. $\pi = 3.1416$ which is a value correct to the fourth decimal place.

Lalla also wrote a commentary on Khandakhadyaka, a work of Brahmagupta. Lalla's commentary has not survived but there is another work on astrology by Lalla which has survived, namely the Jyotisaratnakosa. This was a very popular work which was the main one on the subject in India for around 300 years.

Govindasvami

Born: about 800 in India

Died: about 860 in India

Govindasvami (or Govindasvamin) was an Indian mathematical astronomer whose most famous treatise was a commentary on the Mahabhaskariya of Bhaskara I.

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Bhaskara I wrote the Mahabhaskariya in about 600 A.D. It is an eight chapter work on Indian mathematical astronomy and includes topics which were fairly standard for such works at this time. It discussed topics such as the longitudes of the planets, conjunctions of the planets with each other and with bright stars, eclipses of the sun and the moon, risings and settings, and the lunar crescent.

Govindasvami wrote the Bhasya in about 830 which was a commentary on the Mahabhaskariya. In Govindasvami's commentary there appear many examples of using a place-value Sanskrit system of numerals. One of the most interesting aspects of the commentary, however, is Govindasvami's construction of a sine table.

Indian mathematicians and astronomers constructed sine table with great precision. They were used to calculate the positions of the planets as accurately as possible so had to be computed with high degrees of accuracy. Govindasvami considered the sexagesimal fractional parts of the twenty-four tabular sine differences from the Aryabhatiya. These lead to more correct sine values at intervals of 90/24 = 3 45'. In the commentary Govindasvami found certain other empirical rules relating to computations of sine differences in the argumental range of 60 to 90 degrees.

Mahavira

Born: about 800 in possibly Mysore, India

Died: about 870 in India

Mahavira (or Mahaviracharya meaning Mahavira the Teacher) was of the Jaina religion and was familiar with Jaina mathematics. He worked in Mysore in southern Indian where he was a member of a school of mathematics. If he was not born in Mysore then it is very likely that he was born close to this town in the same region of India. We have essentially no other biographical details although we can

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gain just a little of his personality from the acknowledgement he gives in the introduction to his only known work, see below. However Jain mentions six other works which he credits to Mahavira and he emphasises the need for further research into identifying the complete list of his works.

The only known book by Mahavira is Ganita Sara Samgraha, dated 850 AD, which was designed as an updating of Brahmagupta's book. Filliozat writes:-

> *This book deals with the teaching of Brahmagupta but contains both simplifications and additional information. ... Although like all Indian versified texts, it is extremely condensed, this work, from a pedagogical point of view, has a significant advantage over earlier texts.*

It consisted of nine chapters and included all mathematical knowledge of mid-ninth century India. It provides us with the bulk of knowledge which we have of Jaina mathematics and it can be seen as in some sense providing an account of the work of those who developed this mathematics. There were many Indian mathematicians before the time of Mahavira but, perhaps surprisingly, their work on mathematics is always contained in texts which discuss other topics such as astronomy. The Ganita Sara Samgraha by Mahavira is the earliest Indian text which we possess which is devoted entirely to mathematics.

In the introduction to the work Mahavira paid tribute to the mathematicians whose work formed the basis of his book. These mathematicians included Aryabhatta I, Bhaskara I, and Brahmagupta. Mahavira writes:-

> *With the help of the accomplished holy sages, who are worthy to be worshipped by the lords of the world ... I glean from the great ocean of the knowledge of numbers a little of its essence, in the manner in which gems are picked from the sea, gold from the stony rock*

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and the pearl from the oyster shell; and I give out according to the power of my intelligence, the Sara Samgraha, a small work on arithmetic, which is however not small in importance.

The nine chapters of the Ganita Sara Samgraha are:

1. Terminology
2. Arithmetical operations
3. Operations involving fractions
4. Miscellaneous operations
5. Operations involving the rule of three
6. Mixed operations
7. Operations relating to the calculations of areas
8. Operations relating to excavations
9. Operations relating to shadows

Throughout the work a place-value system with nine numerals is used or sometimes Sanskrit numeral symbols are used. Of interest in Chapter 1 regarding the development of a place-value number system is Mahavira's description of the number 12345654321 which he obtains after a calculation. He describes the number as:-

... beginning with one which then grows until it reaches six, then decreases in reverse order.

Notice that this wording makes sense to us using a place-value system but would not make sense in other systems. It is a clear indication that Mahavira is at home with the place-value number system.

Among topics Mahavira discussed in his treatise was operations with fractions including methods to decompose integers and fractions into unit fractions. For example:

2/17 = 1/12 + 1/51 + 1/68.

He examined methods of squaring numbers which, although a special case of multiplying two numbers, can be

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computed using special methods. He also discussed integer solutions of first degree indeterminate equation by a method called kuttaka. The kuttaka (or the "pulveriser") method is based on the use of the Euclidean algorithm but the method of solution also resembles the continued fraction process of Euler given in 1764. The work kuttaka, which occurs in many of the treatises of Indian mathematicians of the classical period, has taken on the more general meaning of "algebra".

An example of a problem given in the Ganita Sara Samgraha which leads to indeterminate linear equations is the following:

> *Three merchants find a purse lying in the road. One merchant says "If I keep the purse, I shall have twice as much money as the two of you together". "Give me the purse and I shall have three times as much" said the second merchant. The third merchant said "I shall be much better off than either of you if I keep the purse, I shall have five times as much as the two of you together". How much money is in the purse? How much money does each merchant have?*

If the first merchant has x, the second y, the third z and p is the amount in the purse then

$$p + x = 2(y + z),\ p + y = 3(x + z),\ p + z = 5(x + y).$$

There is no unique solution but the smallest solution in positive integers is $p = 15$, $x = 1$, $y = 3$, $z = 5$. Any solution in positive integers is a multiple of this solution as Mahavira claims.

Mahavira gave special rules for the use of permutations and combinations which was a topic of special interest in Jaina mathematics. He also described a process for calculating the volume of a sphere and one for calculating the cube root of a number. He looked at some geometrical results including right-angled triangles with rational sides.

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Mahavira also attempts to solve certain mathematical problems which had not been studied by other Indian mathematicians. For example, he gave an approximate formula for the area and the perimeter of an ellipse. Hayashi writes:-

> *The formulas for a conch-like figure have so far been found only in the works of Mahavira and Narayana.*

It is reasonable to ask what a "conch-like figure" is. It is two unequal semicircles (with diameters AB and BC) stuck together along their diameters. Although it might be reasonable to suppose that the perimeter might be obtained by considering the semicircles, Hayashi claims that the formulae obtained:-

> *... were most probably obtained not from the two semicircles AB and BC.*

Prthudakasvami

Born: about 830 in India

Died: about 890 in India

Prthudakasvami is best known for his work on solving equations.

The solution of a first-degree indeterminate equation by a method called kuttaka (or "pulveriser") was given by Aryabhatta I. This method of finding integer solutions resembles the continued fraction process and can also be seen as a use of the Euclidean algorithm.

Brahmagupta seems to have used a method involving continued fractions to find integer solutions of an indeterminate equation of the type ax + c = by. Prthudakasvami's commentary on Brahmagupta's work is helpful in showing how "algebra", that is the method of calculating with the unknown, was developing in India. Prthudakasvami discussed the kuttaka method which he

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renamed as "bijagnita" which means the method of calculating with unknown elements.

To see just how this new idea of algebra was developing in India, we look at the notation which was being used by Prthudakasvami in his commentary on Brahmagupta's Brahma Sputa Siddhanta. In this commentary Prthudakasvami writes the equation $10x + 8 = x^2 + 1$ as:

yava 0 ya 10 ru 8

yava 1 ya 0 ru 1

Here yava is an abbreviation for yavat avad varga which means the "square of the unknown quantity", ya is an abbreviation for yavat havat which means the "unknown quantity", and ru is an abbreviation for rupa which means "constant term". Hence the top row reads

$0x^2 + 10x + 8$

while the second row reads

$x^2 + 0x + 1$

The whole equation is therefore

$0x^2 + 10x + 8 = x^2 + 0x + 1$

or

$10x + 8 = x^2 + 1.$

Shankara Narayana

Born: about 840 in India

Died: about 900 in India

Shankara Narayana (or Shankaranarayana) was an Indian astronomer and mathematician. He was a disciple of the astronomer and mathematician Govindasvami. His most famous work was the Laghubhaskariyavivarana which was a commentary on the Laghubhaskariya of Bhaskara I which in turn is based on the work of Aryabhatta I.

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The Laghubhaskariyavivarana was written by Shankara Narayana in 869 AD for the author writes in the text that it is written in the Shaka year 791 which translates to a date AD by adding 78. It is a text which covers the standard mathematical methods of Aryabhatta I such as the solution of the indeterminate equation by = ax c (a, b, c integers) in integers which is then applied to astronomical problems. The standard Indian method involves using the Euclidean algorithm. It is called kuttakara ("pulveriser") but the term eventually came to have a more general meaning like "algebra". The paper examines this method. The reader who is wondering what the determination of "mati" means in the title of the paper then it refers to the optional number in a guessed solution and it is a feature which differs from the original method as presented by Bhaskara I.

Perhaps the most unusual feature of the Laghubhaskariyavivarana is the use of katapayadi numeration as well as the place-value Sanskrit numerals which Shankara Narayana frequently uses. Shankara Narayana is the first author known to use katapayadi numeration with this name but he did not invent it for it appears to be identical to a system invented earlier which was called varnasamjna. The numeration system varnasamjna was almost certainly invented by the astronomer Haridatta, and it was explained by him in a text which many historians believe was written in 684 but this would contradict what Shankara Narayana himself writes. This point is discussed below. First we should explain ideas behind Shankara Narayana's katapayadi numeration.

The system is based on writing numbers using the letters of the Indian alphabet. Let us quote from:-

... the numerical attribution of syllables corresponds to the following rule, according to the regular order of succession of the letters of the Indian alphabet: the first nine letters represent the numbers 1 to 9 while

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the tenth corresponds to zero; the following nine letters also receive the values 1 to 9 whilst the following letter has the value zero; the next five represent the first five units; and the last eight represent the numbers 1 to 8.

Under this system 1 to 5 are represented by four different letters. For example 1 is represented by the letters ka, ta, pa, ya which give the system its name (ka, ta, pa, ya becomes katapaya). Then 6, 7, 8 are represented by three letters and finally nine and zero are represented by two letters.

The system was a spoken one in the sense that consonants and vowels which are not vocalised have no numerical value. The system is a place-value system with zero but one may reasonably ask why such an apparently complicated numeral system might ever come to be invented. Well the answer must be that it lead to easily remembered mnemonics. In fact many different "words" could represent the same number and this was highly useful for works written in verse as the Indian texts tended to be.

Let us return to the interesting point about the date of Haridatta. Very unusually for an Indian text, Shankara Narayana expresses his thanks to those who have gone before him and developed the ideas about which he is writing. This in itself is not so unusual but the surprise here is that Shankara Narayana claims to give the list in chronological order. His list is

Aryabhatta I

Varahamihira

Bhaskara I

Govindasvami

Haridatta

[Note that we have written Bhaskara I where Shankara Narayana simply wrote Bhaskara. The more famous

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Bhaskara II lived nearly 300 years after Shankara Narayana.]

The chronological order in the list agrees with the dates we have for the first four of these mathematicians. However, putting Haridatta after Govindasvami would seem an unlikely mistake for Shankara Narayana to make if Haridatta really did write his text in 684 since Shankara Narayana was himself a disciple of Govindasvami. If the dating given by Shankara Narayana is correct then katapayadi numeration had been invented only a few years before he wrote his text.

Sridhara

Born: about 870 in possibly Bengal, India

Died: about 930 in India

Sridhara is now believed to have lived in the ninth and tenth centuries. However, there has been much dispute over his date and in different works the dates of the life of Sridhara have been placed from the seventh century to the eleventh century. The best present estimate is that he wrote around 900 AD, a date which is deduced from seeing which other pieces of mathematics he was familiar with and also seeing which later mathematicians were familiar with his work. We do know that Sridhara was a Hindu but little else is known. Two theories exist concerning his birthplace which are far apart. Some historians give Bengal as the place of his birth while other historians believe that Sridhara was born in southern India.

Sridhara is known as the author of two mathematical treatises, namely the Trisatika (sometimes called the Patiganitasara) and the Patiganita. However at least three other works have been attributed to him, namely the Bijaganita, Navasati, and Brhatpati. Information about these books was given the works of Bhaskara II (writing around 1150), Makkibhatta (writing in 1377), and Raghavabhatta

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(writing in 1493). We give details below of Sridhara's rule for solving quadratic equations as given by Bhaskara II. There is another mathematical treatise Ganitapancavimsi which some historians believe was written by Sridhara. Hayashi, however, argues that Sridhara is unlikely to have been the author of this work in its present form.

The Patiganita is written in verse form. The book begins by giving tables of monetary and metrological units. Following this algorithms are given for carrying out the elementary arithmetical operations, squaring, cubing, and square and cube root extraction, carried out with natural numbers. Through the whole book Sridhara gives methods to solve problems in terse rules in verse form which was the typical style of Indian texts at this time. All the algorithms to carry out arithmetical operations are presented in this way and no proofs are given. Indeed there is no suggestion that Sridhara realised that proofs are in any way necessary. Often after stating a rule Sridhara gives one or more numerical examples, but he does not give solutions to these example nor does he even give answers in this work.

After giving the rules for computing with natural numbers, Sridhara gives rules for operating with rational fractions. He gives a wide variety of applications including problems involving ratios, barter, simple interest, mixtures, purchase and sale, rates of travel, wages, and filling of cisterns. Some of the examples are decidedly non-trivial and one has to consider this as a really advanced work. Other topics covered by the author include the rule for calculating the number of combinations of n things taken m at a time. There are sections of the book devoted to arithmetic and geometric progressions, including progressions with a fractional numbers of terms, and formulae for the sum of certain finite series are given.

The book ends by giving rules, some of which are only approximate, for the areas of a some plane polygons. In fact

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the text breaks off at this point but it certainly was not the end of the book which is missing in the only copy of the work which has survived.

We do know something of the missing part, however, for the Patiganitasara is a summary of the Patiganita including the missing portion.

Shukla examines Sridhara's method for finding rational solutions of $Nx^2 \pm 1 = y^2$, $1 - Nx^2 = y^2$, $Nx^2 \pm C = y^2$, and $C - Nx^2 = y^2$ which Sridhara gives in the Patiganita.

Shukla states that the rules given there are different from those given by other Hindu mathematicians.

Sridhara was one of the first mathematicians to give a rule to solve a quadratic equation. Unfortunately, as we indicated above, the original is lost and we have to rely on a quotation of Sridhara's rule from Bhaskara II:-

> *Multiply both sides of the equation by a known quantity equal to four times the coefficient of the square of the unknown; add to both sides a known quantity equal to the square of the coefficient of the unknown; then take the square root.*

To see what this means take

$ax^2 + bx = c.$

Multiply both sides by 4a to get

$4a^2x^2 + 4abx = 4ac$

then add b2 to both sides to get

$4a^2x^2 + 4abx + b^2 = 4ac + b^2$

and, taking the square root

$2ax + b = v(4ac + b^2).$

There is no suggestion that Sridhara took two values when he took the square root.

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Aryabhatta II

Born: about 920 in India

Died: about 1000 in India

Essentially nothing is known of the life of Aryabhatta II. Historians have argued about his date and have come up with many different theories. Pingree gives the date for his main publications as being between 950 and 1100. This is deduced from the usual arguments such as which authors Aryabhatta II refers to and which refer to him. G R Kaye argued in 1910 that Aryabhatta II lived before al-Biruni but Datta in 1926 showed that these dates were too early.

The article argues for a date of about 950 for Aryabhatta II's main work, the Mahasiddhanta, but R Billiard has proposed a date for Aryabhatta II in the sixteenth century. Most modern historians, however, consider the most likely dates for his main work as around 950 and we have given very approximate dates for his birth and death based on this hypothesis.

The most famous work by Aryabhatta II is the Mahasiddhanta which consists of eighteen chapters. The treatise is written in Sanskrit verse and the first twelve chapters form a treatise on mathematical astronomy covering the usual topics that Indian mathematicians worked on during this period. The topics included in these twelve chapters are: the longitudes of the planets, eclipses of the sun and moon, the projection of eclipses, the lunar crescent, the rising and setting of the planets, conjunctions of the planets with each other and with the stars.

The remaining six chapters of the Mahasiddhanta form a separate part entitled On the sphere. It discusses topics such as geometry, geography and algebra with applications to the longitudes of the planets.

In Mahasiddhanta Aryabhatta II gives in about twenty verses detailed rules to solve the indeterminate equation:

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by = ax + c. The rules apply in a number of different cases such as when c is positive, when c is negative, when the number of the quotients of the mutual divisions is even, when this number of quotients is odd, etc.

Aryabhatta II also gave a method to calculate the cube root of a number, but his method was not new, being based on that given many years earlier by Aryabhatta I.

Aryabhatta II constructed a sine table correct up to five decimal places when measured in decimal parts of the radius. Indian mathematicians were very interested in giving accurate sine tables since they were used to calculate the planetary positions as accurately as possible.

Vijayanandi

Born: about 940 in Benares (now Varanasi), India

Died: about 1010 in India

Vijayanandi (or Vijayanandin) was the son of Jayananda. He was born into the Brahman caste which meant he was from the highest ranking caste of Hindu priests. He was an Indian mathematician and astronomer whose most famous work was the Karanatilaka.

We should note that there was another astronomer named Vijayanandi who was mentioned by Varahamihira in one of his works. Since Varahamihira wrote around 550 and the Karanatilaka was written around 966, there must be two astronomers both named "Vijayanandi".

The Karanatilaka has not survived in its original form but we know of the text through an Arabic translation by al-Biruni. It is a work in fourteen chapters covering the standard topics of Indian astronomy.

It deals with the topics of: units of time measurement; mean and true longitudes of the sun and moon; the length of daylight; mean longitudes of the five planets; true

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longitudes of the five planets; the three problems of diurnal rotation; lunar eclipses, solar eclipses; the projection of eclipses; first visibility of the planets; conjunctions of the planets with each other and with fixed stars; the moon's crescent; and the patas of the moon and sun.

The Indians had a cosmology which was based on long periods of time with astronomical events occurring a certain whole number of times within the cycles. This system led to much work on integer solutions of equations and their application to astronomy.

In particular there was, according to Aryabhatta I, a basic period of 4320000 years called a mahayuga and it was assumed that the sun, the moon, their apsis and node, and the planets reached perfect conjunctions after this period. Hence it was assumed that the periods were rational multiples of each other.

All the planets and the node and apsis of the moon and sun had to have an integer number of revolutions in the mahayuga. Many Indian astronomers proposed different values for these integral numbers of revolutions.

For the number of revolutions of the apsis and node of the moon per mahayuga, Aryabhatta I proposed 488219 and 232226, respectively.

However Vijayanandi changed these numbers to 488211 and 232234. The reasons for giving the new numbers is unclear. Like other Indian astronomers, Vijayanandi made contributions to trigonometry and it appears that his calculation of the periods was computed by using tables of sines and versed sines.

It is significant that accuracy was need in trigonometric tables to give accurate astronomical theories and this motivated many of the Indian mathematicians to produce more accurate methods of approximating entries in tables.

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Sripati

Born: 1019 in (probably) Rohinikhanda, Maharashtra, India

Died: 1066 in India

Sripati's father was Nagadeva (sometimes written as Namadeva) and Nagadeva's father, Sripati's paternal grandfather, was Kesava. Sripati was a follower of the teaching of Lalla writing on astrology, astronomy and mathematics. His mathematical work was undertaken with applications to astronomy in mind, for example a study of spheres. His work on astronomy was undertaken to provide a basis for his astrology. Sripati was the most prominent Indian mathematicians of the 11th Century.

Among Sripati's works are: Dhikotidakarana written in 1039, a work of twenty verses on solar and lunar eclipses; Dhruvamanasa written in 1056, a work of 105 verses on calculating planetary longitudes, eclipses and planetary transits; Siddhantasekhara a major work on astronomy in 19 chapters; and Ganitatilaka an incomplete arithmetical treatise in 125 verses based on a work by Sridhara.

The titles of Chapters 13, 14, and 15 of the Siddhantasekhara are Arithmetic, Algebra and On the Sphere. Chapter 13 consists of 55 verses on arithmetic, mensuration, and shadow reckoning. It is probable that the lost portion of the arithmetic treatise Ganitatilaka consisted essentially of verses 19-55 of this chapter. The 37 verses of Chapter 14 on algebra state various rules of algebra without proof. These are given in verbal form without algebraic symbols. In verses 3, 4 and 5 of this chapter Sripati gave the rules of signs for addition, subtraction, multiplication, division, square, square root, cube and cube root of positive and negative quantities. His work on equations in this chapter contains the rule for solving a quadratic equation and, more impressively, he gives the identity:

$$O(x + Oy) = O[(x + O(x^2 - y)]/2 + O[(x - O(x^2 - y)]/2)$$

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Other mathematics included in Sripati's work includes, in particular, rules for the solution of simultaneous indeterminate equations of the first degree that are similar to those given by Brahmagupta.

Sripati obtained more fame in astrology than in other areas and it is fair to say that he considered this to be his most important contributions. He wrote the Jyotisaratnamala which was an astrology text in twenty chapters based on the Jyotisaratnakosa of Lalla.

Sripati wrote a commentary on this work in Marathi and it is one of the oldest works to have survived that is written in that language. Marathi is the oldest of the regional languages in Indo-Aryan, dating from about 1000.

Another work on astrology written by Sripati is the Jatakapaddhati or Sripatipaddhati which is in eight chapters :-

> *... one of the fundamental textbooks for later Indian genethlialogy, contributing an impressive elaboration to the computation of the strengths of the planets and astrological places. It was enormously popular, as the large number of manuscripts, commentaries, and imitations attests.*

Genethlialogy was the science of casting nativities and it was the earliest branch of astrology which claimed to be able to predict the course of a person's life based on the positions of the planets and of the signs of the zodiac at the moment the person was born or conceived.

There is one other work on astrology the Daivajnavallabha which some historians claim was written by Sripati while other claim that it is the work of Varahamihira. As yet nobody has come up with a definite case to show which of these two is the author, or even whether the author is another astrologer.

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Brahmadeva

Born: about 1060 in (possibly) Mathura, India

Died: about 1130 in India

Brahmadeva was the son of Candrabudha. The family came from the Mathura district of Uttar Pradesh in northern India. He was born into the Brahman caste which meant he was from the highest ranking caste of Hindu priests.

The Brahman caste had a strong tradition of education so Brahmadeva would have received one of the best educations of men of his time.

We have only one work by Brahmadeva and this is Karanaprakasa which is a commentary on the Aryabhatiya by Aryabhatta I.

Brahmadeva's work is in nine chapters and it follows the contents of the original Aryabhatiya.

Topics covered include the longitudes of the planets, problems relating to the daily rotation of the heavens, eclipses of the sun and the moon, risings and settings, the lunar crescent, and conjunctions of the planets.

The work contains some contributions to trigonometry, motivated by its application to mathematical astronomy. It is this aspect of the work which is mentioned by Gupta.

Different commentaries on the Aryabhatiya achieved popularity in different parts of India. Brahmadeva's commentary seems to have been particularly popular in Madras, Mysore and Maharastra.

The more important commentaries on the Aryabhatiya became the basis for further commentaries and indeed this is what happened to the Karanaprakasa.

Commentaries on Brahmadeva's work continued to appear up to the seventeenth century.

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Bhaskara

Born: 1114 in Vijayapura, India

Died: 1185 in Ujjain, India

Bhaskara is also known as Bhaskara II or as Bhaskaracharya, this latter name meaning "Bhaskara the Teacher". Since he is known in India as Bhaskaracharya we will refer to him throughout this article by that name. Bhaskaracharya's father was a Brahman named Mahesvara. Mahesvara himself was famed as an astrologer. This happened frequently in Indian society with generations of a family being excellent mathematicians and often acting as teachers to other family members.

Bhaskaracharya became head of the astronomical observatory at Ujjain, the leading mathematical centre in India at that time. Outstanding mathematicians such as Varahamihira and Brahmagupta had worked there and built up a strong school of mathematical astronomy.

In many ways Bhaskaracharya represents the peak of mathematical knowledge in the 12th century. He reached an understanding of the number systems and solving equations which was not to be achieved in Europe for several centuries.

Six works by Bhaskaracharya are known but a seventh work, which is claimed to be by him, is thought by many historians to be a late forgery. The six works are: Lilavati (The Beautiful) which is on mathematics; Bijaganita (Seed Counting or Root Extraction) which is on algebra; the Siddhantasiromani which is in two parts, the first on mathematical astronomy with the second part on the sphere; the Vasanabhasya of Mitaksara which is Bhaskaracharya's own commentary on the Siddhantasiromani; the Karanakutuhala (Calculation of Astronomical Wonders) or Brahmatulya which is a simplified version of the Siddhantasiromani ; and the Vivarana which is a commentary

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on the Shishyadhividdhidatantra of Lalla. It is the first three of these works which are the most interesting, certainly from the point of view of mathematics, and we will concentrate on the contents of these.

Given that he was building on the knowledge and understanding of Brahmagupta it is not surprising that Bhaskaracharya understood about zero and negative numbers. However his understanding went further even than that of Brahmagupta. To give some examples before we examine his work in a little more detail we note that he knew that $x^2 = 9$ had two solutions.

He also gave the formula:

$$\sqrt{a \pm \sqrt{b}} = \sqrt{\frac{a + \sqrt{a^2 - b}}{2}} \pm \sqrt{\frac{a - \sqrt{a^2 - b}}{2}}$$

Bhaskaracharya studied Pell's equation $px^2 + 1 = y^2$ for $p = 8, 11, 32, 61$ and 67. When $p = 61$ he found the solutions $x = 226153980$, $y = 1776319049$. When $p = 67$ he found the solutions $x = 5967$, $y = 48842$. He studied many Diophantine problems.

Let us first examine the Lilavati. First it is worth repeating the story told by Fyzi who translated this work into Persian in 1587. We give the story as given by Joseph:-

Lilavati was the name of Bhaskaracharya's daughter. From casting her horoscope, he discovered that the auspicious time for her wedding would be a particular hour on a certain day. He placed a cup with a small hole at the bottom of the vessel filled with water, arranged so that the cup would sink at the beginning of the propitious hour. When everything was ready and the cup was placed in the vessel, Lilavati suddenly out of curiosity bent over the vessel and a pearl from her dress fell into the cup and blocked the hole in it. The lucky hour passed without the cup sinking.

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Bhaskaracharya believed that the way to console his dejected daughter, who now would never get married, was to write her a manual of mathematics!

This is a charming story but it is hard to see that there is any evidence for it being true. It is not even certain that Lilavati was Bhaskaracharya's daughter. There is also a theory that Lilavati was Bhaskaracharya's wife. The topics covered in the thirteen chapters of the book are: definitions; arithmetical terms; interest; arithmetical and geometrical progressions; plane geometry; solid geometry; the shadow of the gnomon; the kuttaka; combinations.

In dealing with numbers Bhaskaracharya, like Brahmagupta before him, handled efficiently arithmetic involving negative numbers. He is sound in addition, subtraction and multiplication involving zero but realised that there were problems with Brahmagupta's ideas of dividing by zero. Madhukar Mallayya argues that the zero used by Bhaskaracharya in his rule (a.0)/0 = a, given in Lilavati, is equivalent to the modern concept of a non-zero "infinitesimal". Although this claim is not without foundation, perhaps it is seeing ideas beyond what Bhaskaracharya intended.

Bhaskaracharya gave two methods of multiplication in his Lilavati. We follow Ifrah who explains these two methods due to Bhaskaracharya. To multiply 325 by 243 Bhaskaracharya writes the numbers thus:

```
243 243 243
3 2 5
-------+------------
```

Now working with the rightmost of the three sums he computed 5 times 3 then 5 times 2 missing out the 5 times 4 which he did last and wrote beneath the others one place to the left. Note that this avoids making the "carry" in ones head.

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```
243 243 243
3 2 5
-------------------
1015
20
-------------------
```

Now add the 1015 and 20 so positioned and write the answer under the second line below the sum next to the left.

```
243 243 243
3 2 5
-------------------
1015
20
-------------------
1215
```

Work out the middle sum as the right-hand one, again avoiding the "carry", and add them writing the answer below the 1215 but displaced one place to the left.

```
243 243 243
3 2 5
-------------------
4 6 1015
8 20
-------------------
1215
486
```

Finally work out the left most sum in the same way and again place the resulting addition one place to the left under the 486.

```
243 243 243
3 2 5
-------------------
6 9 4 6 1015
12 8 20
-------------------
```

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1215
486
729

Finally add the three numbers below the second line to obtain the answer 78975.

243 243 243
3 2 5
---
6 9 4 6 1015
12 8 20

1215
486
729

78975

Despite avoiding the "carry" in the first stages, of course one is still faced with the "carry" in this final addition.

The second of Bhaskaracharya's methods proceeds as follows:

325
243

Multiply the bottom number by the top number starting with the left-most digit and proceeding towards the right. Displace each row one place to start one place further right than the previous line.

First step

325
243

729

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Second step

```
 325
 243
--------
 729
 486
```

Third step, then add

```
 325
 243
--------
 729
 486
 1215
--------
 78975
```

Bhaskaracharya, like many of the Indian mathematicians, considered squaring of numbers as special cases of multiplication which deserved special methods. He gave four such methods of squaring in Lilavati.

Here is an example of explanation of inverse proportion taken from Chapter 3 of the Lilavati. Bhaskaracharya writes:-

In the inverse method, the operation is reversed. That is the fruit to be multiplied by the augment and divided by the demand. When fruit increases or decreases, as the demand is augmented or diminished, the direct rule is used. Else the inverse.

Rule of Three Inverse: If the fruit diminish as the requisition increases, or augment as that decreases, they, who are skilled in accounts, consider the rule of three to be inverted. When there is a diminution of fruit, if there be increase of requisition, and increase of fruit if there be diminution of requisition, then the inverse rule of three is employed.

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As well as the rule of three, Bhaskaracharya discusses examples to illustrate rules of compound proportions, such as the rule of five (Pancarasika), the rule of seven (Saptarasika), the rule of nine (Navarasika), etc. Bhaskaracharya's examples of using these rules are discussed.

An example from Chapter 5 on arithmetical and geometrical progressions is the following:-

Example: On an expedition to seize his enemy's elephants, a king marched two yojanas the first day. Say, intelligent calculator, with what increasing rate of daily march did he proceed, since he reached his foe's city, a distance of eighty yojanas, in a week?

Bhaskaracharya shows that each day he must travel $\frac{22}{7}$ yojanas further than the previous day to reach his foe's city in 7 days.

An example from Chapter 12 on the kuttaka method of solving indeterminate equations is the following:-

Example: Say quickly, mathematician, what is that multiplier, by which two hundred and twenty-one being multiplied, and sixty-five added to the product, the sum divided by a hundred and ninety-five becomes exhausted.

Bhaskaracharya is finding integer solution to $195x = 221y + 65$. He obtains the solutions $(x,y) = (6,5)$ or $(23,20)$ or $(40, 35)$ and so on.

In the final chapter on combinations Bhaskaracharya considers the following problem. Let an n-digit number be represented in the usual decimal form as

(*) $d_1 d_2 \ldots d_n$

where each digit satisfies $1 \le d_j \le 9$, $j = 1, 2, \ldots, n$. Then Bhaskaracharya's problem is to find the total number of numbers of the form (*) that satisfy

$$d_1 + d_2 + \ldots + d_n = S.$$

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In his conclusion to Lilavati Bhaskaracharya writes:-

Joy and happiness is indeed ever increasing in this world for those who have Lilavati clasped to their throats, decorated as the members are with neat reduction of fractions, multiplication and involution, pure and perfect as are the solutions, and tasteful as is the speech which is exemplified.

The Bijaganita is a work in twelve chapters. The topics are: positive and negative numbers; zero; the unknown; surds; the kuttaka; indeterminate quadratic equations; simple equations; quadratic equations; equations with more than one unknown; quadratic equations with more than one unknown; operations with products of several unknowns; and the author and his work.

Having explained how to do arithmetic with negative numbers, Bhaskaracharya gives problems to test the abilities of the reader on calculating with negative and affirmative quantities:-

Example: Tell quickly the result of the numbers three and four, negative or affirmative, taken together; that is, affirmative and negative, or both negative or both affirmative, as separate instances; if thou know the addition of affirmative and negative quantities.

Negative numbers are denoted by placing a dot above them: The characters, denoting the quantities known and unknown, should be first written to indicate them generally; and those, which become negative should be then marked with a dot over them.

Example: Subtracting two from three, affirmative from affirmative, and negative from negative, or the contrary, tell me quickly the result ...

In Bijaganita Bhaskaracharya attempted to improve on Brahmagupta's attempt to divide by zero (and his own description in Lilavati) when he wrote:-

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A quantity divided by zero becomes a fraction the denominator of which is zero. This fraction is termed an infinite quantity. In this quantity consisting of that which has zero for its divisor, there is no alteration, though many may be inserted or extracted; as no change takes place in the infinite and immutable God when worlds are created or destroyed, though numerous orders of beings are absorbed or put forth.

So Bhaskaracharya tried to solve the problem by writing n/0 = 8. At first sight we might be tempted to believe that Bhaskaracharya has it correct, but of course he does not. If this were true then 0 times 8 must be equal to every number n, so all numbers are equal. The Indian mathematicians could not bring themselves to the point of admitting that one could not divide by zero.

Equations leading to more than one solution are given by Bhaskaracharya:-

Example: Inside a forest, a number of apes equal to the square of one-eighth of the total apes in the pack are playing noisy games. The remaining twelve apes, who are of a more serious disposition, are on a nearby hill and irritated by the shrieks coming from the forest. What is the total number of apes in the pack?

The problem leads to a quadratic equation and Bhaskaracharya says that the two solutions, namely 16 and 48, are equally admissible.

The kuttaka method to solve indeterminate equations is applied to equations with three unknowns.

The problem is to find integer solutions to an equation of the form ax + by + cz = d. An example he gives is:-

Example: The horses belonging to four men are 5, 3, 6 and 8. The camels belonging to the same men are 2, 7, 4 and 1. The mules belonging to them are 8, 2, 1 and 3 and

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the oxen are 7, 1, 2 and 1, all four men have equal fortunes. Tell me quickly the price of each horse, camel, mule and ox.

Of course such problems do not have a unique solution as Bhaskaracharya is fully aware. He finds one solution, which is the minimum, namely horses 85, camels 76, mules 31 and oxen 4.

Bhaskaracharya's conclusion to the Bijaganita is fascinating for the insight it gives us into the mind of this great mathematician:-

> *A morsel of tuition conveys knowledge to a comprehensive mind; and having reached it, expands of its own impulse, as oil poured upon water, as a secret entrusted to the vile, as alms bestowed upon the worthy, however little, so does knowledge infused into a wise mind spread by intrinsic force.*

It is apparent to men of clear understanding, that the rule of three terms constitutes arithmetic and sagacity constitutes algebra. Accordingly I have said ... The rule of three terms is arithmetic; spotless understanding is algebra. What is there unknown to the intelligent? Therefore for the dull alone it is set forth.

The Siddhantasiromani is a mathematical astronomy text similar in layout to many other Indian astronomy texts of this and earlier periods. The twelve chapters of the first part cover topics such as: mean longitudes of the planets; true longitudes of the planets; the three problems of diurnal rotation; syzygies; lunar eclipses; solar eclipses; latitudes of the planets; risings and settings; the moon's crescent; conjunctions of the planets with each other; conjunctions of the planets with the fixed stars; and the patas of the sun and moon.

The second part contains thirteen chapters on the sphere. It covers topics such as: praise of study of the sphere; nature of the sphere; cosmography and geography; planetary mean

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motion; eccentric epicyclic model of the planets; the armillary sphere; spherical trigonometry; ellipse calculations; first visibilities of the planets; calculating the lunar crescent; astronomical instruments; the seasons; and problems of astronomical calculations.

There are interesting results on trigonometry in this work. In particular Bhaskaracharya seems more interested in trigonometry for its own sake than his predecessors who saw it only as a tool for calculation. Among the many interesting results given by Bhaskaracharya are:

$$\sin(a + b) = \sin a \cos b + \cos a \sin b$$

and

$$\sin(a - b) = \sin a \cos b - \cos a \sin b.$$

Bhaskaracharya rightly achieved an outstanding reputation for his remarkable contribution. In 1207 an educational institution was set up to study Bhaskaracharya's works. A medieval inscription in an Indian temple reads:-

Triumphant is the illustrious Bhaskaracharya whose feats are revered by both the wise and the learned. A poet endowed with fame and religious merit, he is like the crest on a peacock.

It is from this quotation that the title of Joseph's book comes.

Mahendra Suri

Born: about 1340 in Western India

Died: about 1410 in India

Mahendra Suri was a Jain. Jainism began around the sixth century BC and the religion had a strong influence on mathematics particularly in the last couple of centuries BC. By the time of Mahendra Suri, however, Jainism had lost support as a national religion and was much less vigorous.

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It had been influenced by Islam and in particular Islamic astronomy came to form a part of the background. However, Pingree writes that this filtering of Islamic astronomy into Indian culture was:-

> *... not allowed to affect in any way the structure of the traditional science.*

Mahendra Suri was a pupil of Madana Suri. He is famed as the first person to write a Sanskrit treatise on the astrolabe. Ohashi writes of the early history of the astrolabe in the Delhi Sultanate in India:-

The astrolabe was introduced into India at the time of Firuz Shah Tughluq (reign AD 1351 - 88), and Mahendra Suri wrote the first Sanskrit treatise on the astrolabe entitled Yantraraja (AD 1370).

The Delhi Sultanate was established around 1200 and from that time on Muslim culture flourished in India. The ideas of Islamic astronomy began to appear in works in the Sanskrit language and it is the Islamic ideas on the astrolabe which Mahendra Suri wrote on in his famous text. It is clear from the various references in the text and also from the particular values that Mahendra Suri uses for the angle of the ecliptic etc. that his work is based on Islamic rather than traditional Indian astronomy works.

Narayana Pandit

Born: about 1340 in India

Died: about 1400 in India

Narayana was the son of Narsimha (sometimes written Narasimha). We know that he wrote his most famous work Ganita Kaumudi on arithmetic in 1356 but little else is known of him. His mathematical writings show that he was strongly influenced by Bhaskara II and he wrote a commentary on the Lilavati of Bhaskara II called Karmapradipika. Some historians dispute that Narayana is

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the author of this commentary which they attribute to Madhava.

In the Ganita Kaumudi Narayana considers the mathematical operation on numbers. Like many other Indian writers of arithmetics before him he considered an algorithm for multiplying numbers and he then looked at the special case of squaring numbers. One of the unusual features of Narayana's work Karmapradipika is that he gave seven methods of squaring numbers which are not found in the work of other Indian mathematicians.

He discussed another standard topic for Indian mathematicians namely that of finding triangles whose sides had integral values. In particular he gave a rule of finding integral triangles whose sides differ by one unit of length and which contain a pair of right-angled triangles having integral sides with a common integral height. In terms of geometry Narayana gave a rule for a segment of a circle. Narayana :-

> *... derived his rule for a segment of a circle from Mahavira's rule for an 'elongated circle' or an ellipse-like figure.*

Narayana also gave a rule to calculate approximate values of a square root. He did this by using an indeterminate equation of the second order, $Nx^2 + 1 = y^2$, where N is the number whose square root is to be calculated. If x and y are a pair of roots of this equation with $x < y$ then vN is approximately equal to y/x. To illustrate this method Narayana takes N = 10. He then finds the solutions x = 6, y = 19 which give the approximation 19/6 = 3.1666666666666666667, which is correct to 2 decimal places. Narayana then gives the solutions x = 228, y = 721 which give the approximation 721/228 = 3.1622807017543859649, correct to four places. Finally Narayana gives the pair of solutions x = 8658, y = 227379 which give the approximation 227379/8658 = 3.1622776622776622777, correct to eight

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decimal places. Note for comparison that v10 is, correct to 20 places, 3.1622776601683793320.

The thirteenth chapter of Ganita Kaumudi was called Net of Numbers and was devoted to number sequences. For example, he discussed some problems concerning arithmetic progressions.

The fourteenth chapter (which is the last one) of Naryana's Ganita Kaumudi contains a detailed discussion of magic squares and similar figures. Narayana gave the rules for the formation of doubly even, even and odd perfect magic squares along with magic triangles, rectangles and circles. He used formulae and rules for the relations between magic squares and arithmetic series. He gave methods for finding "the horizontal difference" and the first term of a magic square whose square's constant and the number of terms are given and he also gave rules for finding "the vertical difference" in the case where this information is given.

Madhava of Sangamagramma

Born: 1350 in Sangamagramma (near Cochin), Kerala, India

Died: 1425 in India

Madhava of Sangamagramma was born near Cochin on the coast in the Kerala state in southwestern India. It is only due to research into Keralese mathematics over the last twenty-five years that the remarkable contributions of Madhava have come to light. Rajagopal and Rangachari put his achievement into context when they write:-

[Madhava] took the decisive step onwards from the finite procedures of ancient mathematics to treat their limit-passage to infinity, which is the kernel of modern classical analysis. All the mathematical writings of Madhava have been lost, although some of his texts on astronomy have

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survived. However his brilliant work in mathematics has been largely discovered by the reports of other Keralese mathematicians such as Nilakantha who lived about 100 years later.

Madhava discovered the series equivalent to the Maclaurin expansions of sin x, cos x, and arctan x around 1400, which is over two hundred years before they were rediscovered in Europe. Details appear in a number of works written by his followers such as Mahajyanayana prakara which means Method of computing the great sines. In fact this work had been claimed by some historians such as Sarma to be by Madhava himself but this seems highly unlikely and it is now accepted by most historians to be a 16th century work by a follower of Madhava.

Jyesthadeva wrote Yukti-Bhasa in Malayalam, the regional language of Kerala, around 1550. In [9] Gupta gives a translation of the text and this is also given and a number of other sources. Jyesthadeva describes Madhava's series as follows:-

> *The first term is the product of the given sine and radius of the desired arc divided by the cosine of the arc. The succeeding terms are obtained by a process of iteration when the first term is repeatedly multiplied by the square of the sine and divided by the square of the cosine. All the terms are then divided by the odd numbers 1, 3, 5, ... The arc is obtained by adding and subtracting respectively the terms of odd rank and those of even rank. It is laid down that the sine of the arc or that of its complement whichever is the smaller should be taken here as the given sine. Otherwise the terms obtained by this above iteration will not tend to the vanishing magnitude.*

This is a remarkable passage describing Madhava's series, but remember that even this passage by Jyesthadeva was written more than 100 years before James Gregory

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rediscovered this series expansion. Perhaps we should write down in modern symbols exactly what the series is that Madhava has found. The first thing to note is that the Indian meaning for sine of q would be written in our notation as r sin q and the Indian cosine of would be r cos q in our notation, where r is the radius. Thus the series is

$$r\ q = r(r \sin q)/1(r \cos q) - r(r \sin q)^3/3r(r \cos q)^3$$

$$+ r(r \sin q)^5/5r(r \cos q)^5 - r(r \sin q)^7/7r(r \cos q)^7 + \ldots$$

putting tan = sin/cos and cancelling r gives

$$q = \tan q - (\tan^3 q)/3 + (\tan^5 q)/5 - \ldots$$

which is equivalent to Gregory's series

$$\tan^{-1} q = q - q^3/3 + q^5/5 - \ldots$$

Now Madhava put $q = \pi/4$ into his series to obtain

$$\pi/4 = 1 - 1/3 + 1/5 - \ldots$$

and he also put $q = \pi/6$ into his series to obtain

$$\pi = O12(1 - 1/(3 \times 3) + 1/(5 \times 32) - 1/(7 \times 33) + \ldots$$

We know that Madhava obtained an approximation for p correct to 11 decimal places when he gave

$$\pi = 3.14159265359$$

which can be obtained from the last of Madhava's series above by taking 21 terms. Gupta gives a translation of the Sanskrit text giving Madhava's approximation of π correct to 11 places.

Perhaps even more impressive is the fact that Madhava gave a remainder term for his series which improved the approximation. He improved the approximation of the series for $\pi/4$ by adding a correction term R_n to obtain

$$p/4 = 1 - 1/3 + 1/5 - \ldots 1/(2n-1) \quad Rn$$

Madhava gave three forms of Rn which improved the approximation, namely:

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$R_n = 1/(4n)$ or

$R_n = n/(4n^2 + 1)$ or

$R_n = (n^2 + 1)/(4n^3 + 5n)$.

There has been a lot of work done in trying to reconstruct how Madhava might have found his correction terms. The most convincing is that they come as the first three convergents of a continued fraction which can itself be derived from the standard Indian approximation to π namely $\frac{62832}{20000}$.

Madhava also gave a table of almost accurate values of half-sine chords for twenty-four arcs drawn at equal intervals in a quarter of a given circle. It is thought that the way that he found these highly accurate tables was to use the equivalent of the series expansions:

$\sin q = q - q^3/3! + q^5/5! - \ldots$

$\cos q = 1 - q^2/2! + q^4/4! - \ldots$

Jyesthadeva in Yukti-Bhasa gave an explanation of how Madhava found his series expansions around 1400 which are equivalent to these modern versions rediscovered by Newton around 1676. Historians have claimed that the method used by Madhava amounts to term by term integration.

Rajagopal's claim that Madhava took the decisive step towards modern classical analysis seems very fair given his remarkable achievements. In the same vein Joseph writes:-

> *We may consider Madhava to have been the founder of mathematical analysis. Some of his discoveries in this field show him to have possessed extraordinary intuition, making him almost the equal of the more recent intuitive genius Srinivasa Ramanujan, who spent his childhood and youth at Kumbakonam, not far from Madhava's birthplace.*

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Parameshvara

Born: about 1370 in Alattur, Kerala, India

Died: about 1460 in India

Parameshvara was an Indian astronomer and mathematician who wrote many commentaries on earlier works as well as making many observations. Although his father has not been identified, we know that Parameshvara was born into a Namputiri Brahmana family who were astrologers and astronomers. The family home was Vatasseri (sometimes called Vatasreni) in the village of Alattur. This village was in Kerala and Parameshvara himself gives its coordinates with respect to Ujjain. This puts it at latitude 10 51' north. It is on the north bank of the river Nila at its mouth.

From Parameshvara's writing we know that Rudra was his teacher, and Nilakantha, who knew Parameshvara personally, tells us that Parameshvara's teachers included Madhava and Narayana. We can be fairly confident that the dates we have given for Parameshvara are roughly correct since he made eclipe observations over a period of 55 years. We will say a little more about these observations below. He played an important part in the remarkable developments in mathematics which took place in Kerala in the late 14th and early part of the 15th century.

The commentaries by Parameshvara on a number of works have been published. For example the Karmadipika is a commentary on the Mahabhaskariyam, an astronomical and mathematical work by Bhaskara I. Parameshvara's commentary on the Laghubhaskariyam of Bhaskara I is given. Munjala wrote the astronomical work Laghumanasam in the year 932 and Parameshvara wrote a commentary. It is a work containing typical topics for Indian mathematical astronomy works of this period: the mean motions of the heavenly bodies; the true motions of the heavenly bodies; miscellaneous mathematical rules; the systems of

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coordinates, direction, place and time; eclipses of the sun and the moon; and the operation for apparent longitude.

Aryabhatta gave a rule for determining the height of a pole from the lengths of its shadows in the Aryabhatiya. Parameshvara gave several illustrative examples of the method in his commentary on the Aryabhatiya.

Like many mathematicians from Kerala, Madhava clearly had a very strong influence on Parameshvara. One can see throughout his work that it is teachings by Madhava which direct much of Parameshvara's mathematical ideas. One of Parameshvara's most remarkable mathematical discoveries, no doubt influenced by Madhava, was a version of the mean value theorem. He states the theorem in his commentary Lilavati Bhasya on Bhaskara II's Lilavati. There are other examples of versions of the mean value theorem in Parameshvara's work which we now consider.

The Siddhantadipika by Parameshvara is a commentary on the commentary of Govindasvami on Bhaskara I's Mahabhaskariya. Parameshvara gives some of his eclipse observations in this work including one made at Navaksetra in 1422 and two made at Gokarna in 1425 and 1430. This work also contains a mean value type formula for inverse interpolation of the sine. It presents a one-point iterative technique for calculating the sine of a given angle. In the Siddhantadipika Parameshvara also gives a more efficient approximation that works using a two-point iterative algorithm which turns out to be essentially the same as the modern secant method.

The expression for the radius of the circle in which a cyclic quadrilateral is inscribed, given in terms of the sides of the quadrilateral, is usually attributed to Lhuilier in 1782. However Parameshvara described the rule 350 years earlier. If the sides of the cyclic quadrilateral are a, b, c and d then the radius r of the circumscribed circle was given by Parameshvara as:

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$r^2 = x/y$ where

$x = (ab + cd)(ac + bd)(ad + bc)$

and $y = (a + b + c - d)(b + c + d - a)(c + d + a - b)(d + a + b - c)$.

The original text by Parameshvara describing the rule is given.

Parameshvara made a series of eclipse observations between 1393 and 1432 which we have referred to above. The last observation which we know he made was in 1445 but Nilakantha quotes a verse by Parameshvara in which he claims to have made observations spanning 55 years. The known observatons by Parameshvara do not quite square with this statement, there being a discrepancy of three years. Although we do not know when Parameshvara died we do know, again from Nilakantha, that the two knew each other personally. Since we have a definite date for Nilakantha's birth of 1444 it is hard to believe that Parameshvara died before 1460.

Using his observations, Parameshvara made revisions of the planetary parameters and, like many other Indian astronomers, he constantly attempted to compare the theoretically computed positions of the planets with those which he actually observed. He was keen to improve the theoretical model to bring it into as close an agreement with observations as possible.

Nilakantha Somayaji

Born: 14 June 1444 in Trkkantiyur (near Tirur), Kerala, India

Died: 1544 in India

Nilakantha was born into a Namputiri Brahmin family which came from South Malabar in Kerala. The Nambudiri is the main caste of Kerala. It is an orthodox caste whose

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members consider themselves descendants of the ancient Vedic religion.

He was born in a house called Kelallur which it is claimed coincides with the present Etamana in the village of Trkkantiyur near Tirur in south India. His father was Jatavedas and the family belonged to the Gargya gotra, which was a Indian caste that prohibits marriage to anyone outside the caste. The family followed the Ashvalayana sutra which was a manual of sacrificial ceremonies in the Rigveda, a collection of Vedic hymns. He worshipped the personified deity Soma who was the "master of plants" and the healer of disease. This explains the name Somayaji which means he was from a family qualified to conduct the Soma ritual.

Nilakantha studied astronomy and Vedanta, one of the six orthodox systems of Indian Hindu philosophy, under the teacher Ravi. He was also taught by Damodra who was the son of Parameshvara. Parameshvara was a famous Indian astronomer and Damodra followed his father's teachings. This led Nilakantha also to become a follower of Parameshvara. A number of texts on mathematical astronomy written by Nilakantha have survived. In all he wrote about ten treatises on astronomy.

The Tantrasamgraha is his major astronomy treatise written in 1501. It consists of 432 Sanskrit verses divided into 8 chapters, and it covers various aspects of Indian astronomy. It is based on the epicyclic and eccentric models of planetary motion. The first two chapters deal with the motions and longitudes of the planets. The third chapter Treatise on shadow deals with various problems related with the sun's position on the celestial sphere, including the relationships of its expressions in the three systems of coordinates, namely ecliptic, equatorial and horizontal coordinates.

The fourth and fifth chapters are Treatise on the lunar eclipse and On the solar eclipse and these two chapters treat

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various aspects of the eclipses of the sun and the moon. The sixth chapter is On vyatipata and deals with the complete deviation of the longitudes of the sun and the moon. The seventh chapter On visibility computation discusses the rising and setting of the moon and planets. The final chapter On elevation of the lunar cusps examines the size of the part of the moon which is illuminated by the sun and gives a graphical representation of it.

The Tantrasamgraha is very important in terms of the mathematics Nilakantha uses. In particular he uses results discovered by Madhava and it is an important source of the remarkable mathematical results which he discovered. However, Nilakantha does not just use Madhava's results, he extends them and improves them.

An anonymous commentary entitled Tantrasangraha-vakhya appeared and, somewhat later in about 1550, Jyesthadeva published a commentary entitled Yuktibhasa that contained proofs of the earlier results by Madhava and Nilakantha. This is quite unusual for an Indian text in giving mathematical proofs.

The series $\pi/4 = 1 - 1/3 + 1/5 - 1/7 + \ldots$ is a special case of the series representation for arctan, namely

$$\tan^{-1}x = x - x^3/3 + x^5/5 - x^7/7 + \ldots$$

It is well known that one simply puts $x = 1$ to obtain the series for p/4. The author of reports on the appearance of these series in the work of Leibniz and James Gregory from the 1670s. The contributions of the two European mathematicians to this series are well known but the results on this series in the work of Madhava nearly three hundred years earlier as presented by Nilakantha in the Tantrasamgraha is also discussed.

Nilakantha derived the series expansion

$$\tan^{-1}x = x - x^3/3 + x^5/5 - x^7/7 + \ldots$$

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by obtaining an approximate expression for an arc of the circumference of a circle and then considering the limit. An interesting feature of his work was his introduction of several additional series for $\pi/4$ that converged more rapidly than

$$\pi/4 = 1 - 1/3 + 1/5 - 1/7 + \ldots .$$

The author of provides a reconstruction of how he may have arrived at these results based on the assumption that he possessed a certain continued fraction representation for the tail series

$$1/(n+2) - 1/(n+4) + 1/(n+6) - 1/(n+8) + \ldots .$$

The Tantrasamgraha is not the only work of Nilakantha of which we have the text. He also wrote Golasara which is written in fifty-six Sanskrit verses and shows how mathematical computations are used to calculate astronomical data. The Siddhanta Darpana is written in thirty-two Sanskrit verses and describes a planetary model. The Candracchayaganita is written in thirty-one Sanskrit verses and explains the computational methods used to calculate the moon's zenith distance.

The head of the Nambudiri caste in Nilakantha's time was Netranarayana and he became Nilakantha's patron for another of his major works, namely the Aryabhatiyabhasya which is a commentary on the Aryabhatiya of Aryabhatta I. In this work Nilakantha refers to two eclipses which he observed, the first on 6 March 1467 and the second on 28 July 1501 at Anantaksetra. Nilakantha also refers in the Aryabhatiyabhasya to other works which he wrote such as the Grahanirnaya on eclipses which have not survived.

Jyesthadeva

Born: about 1500 in Kerala, India

Died: about 1575 in Kerala, India

Jyesthadeva lived on the southwest coast of India in the district of Kerala. He belonged to the Kerala school of

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mathematics built on the work of Madhava, Nilakantha Somayaji, Parameshvara and others.

Jyesthadeva wrote a famous text Yuktibhasa which he wrote in Malayalam, the regional language of Kerala. The work is a survey of Kerala mathematics and, very unusually for an Indian mathematical text, it contains proofs of the theorems and gives derivations of the rules it contains. It is one of the main astronomical and mathematical texts produced by the Kerala school. The work was based mainly on the Tantrasamgraha of Nilakantha.

The Yuktibhasa is a major treatise, half on astronomy and half on mathematics, written in 1501. The Tantrasamgraha on which it is based consists of 432 Sanskrit verses divided into 8 chapters, and it covers various aspects of Indian astronomy. It is based on the epicyclic and eccentric models of planetary motion.

The first two chapters deal with the motions and longitudes of the planets. The third chapter Treatise on shadow deals with various problems related with the sun's position on the celestial sphere, including the relationships of its expressions in the three systems of coordinates, namely ecliptic, equatorial and horizontal coordinates.

The fourth and fifth chapters are Treatise on the lunar eclipse and On the solar eclipse and these two chapters treat various aspects of the eclipses of the sun and the moon. The sixth chapter is On vyatipata and deals with the complete deviation of the longitudes of the sun and the moon. The seventh chapter On visibility computation discusses the rising and setting of the moon and planets. The final chapter On elevation of the lunar cusps examines the size of the part of the moon which is illuminated by the sun and gives a graphical representation of it.

The Yuktibhasa is very important in terms of the mathematics Jyesthadeva presents. In particular he presents

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results discovered by Madhava and the treatise is an important source of the remarkable mathematical theorems which Madhava discovered. Written in about 1550, Jyesthadeva's commentary contained proofs of the earlier results by Madhava and Nilakantha which these earlier authors did not give. Gupta gives a translation of the text and this is also given and a number of other sources. Jyesthadeva describes Madhava's series as follows:-

The first term is the product of the given sine and radius of the desired arc divided by the cosine of the arc. The succeeding terms are obtained by a process of iteration when the first term is repeatedly multiplied by the square of the sine and divided by the square of the cosine. All the terms are then divided by the odd numbers 1, 3, 5, ... The arc is obtained by adding and subtracting respectively the terms of odd rank and those of even rank. It is laid down that the sine of the arc or that of its complement whichever is the smaller should be taken here as the given sine. Otherwise the terms obtained by this above iteration will not tend to the vanishing magnitude.

This is a remarkable passage describing Madhava's series, but remember that even this passage by Jyesthadeva was written more than 100 years before James Gregory rediscovered this series expansion. To see how this description of the series fits with Gregory's series for arctan(x) see the biography of Madhava. Other mathematical results presented by Jyesthadeva include topics studied by earlier Indian mathematicians such as integer solutions of systems of first degree equation solved by the kuttaka method, and rules of finding the sines and the cosines of the sum and difference of two angles.

Not only does the mathematics anticipate work by European mathematicians a century later, but the planetary theory presented by Jyesthadeva is similar to that adopted by Tycho Brahe.

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Kamalakara

Born: about 1616 in Benares (now Varanasi), India

Died: about 1700 in India

Kamalakara was an Indian astronomer and mathematician who came from a family of famous astronomers. Kamalakara's father was Narsimha who was born in 1586. Two of Kamalakara's three brothers were also famous astronomer/mathematicians, these being Divakara, who was the eldest of the brothers born in 1606, and Ranganatha who was younger than Kamalakara.

As was common throughout the classical period of Indian mathematics, members of the family acted as teachers to other family members. In particular Kamalakara was taught by his elder brother Divakara while Divakara himself had been taught by their uncle Shiva. Pingree writes :-

> *[Kamalakara] combined traditional Indian astronomy with Aristotelian physics and Ptolemaic astronomy as presented by Islamic scientists (especially Ulugh Beg). Following his family's tradition he wrote a commentary, Manorama, on Ganesa's Grahalaghava and, like his father, Narsimha, another commentary on the Suryasiddhanta, called the Vasanabhasya ...*

Kamalakara's most famous work, the Siddhanta-tattva-viveka, was commented on by Kamalakara himself. The work was completed in 1658. It is a work of fifteen chapters covering standard topics for Indian astronomy texts at this time. It deals with the topics of: units of time measurement; mean motions of the planets; true longitudes of the planets; the three problems of diurnal rotation; diameters and distances of the planets; the earth's shadow; the moon's crescent; risings and settings; syzygies; lunar eclipses, solar eclipses; planetary transits across the sun's disk; the patas of the moon and sun; the "great problems"; and a final chapter which forms a conclusion.

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The third chapter of the Siddhanta-tattva-viveka contains some of the most interesting mathematical results. In that chapter Kamalakara used the addition and subtraction theorems for the sine and the cosine to give trigonometric formulae for the sines and cosines of double, triple, quadruple and quintuple angles. In particular he gives formulae for sin(A/2) and sin(A/4) in terms of sin(A) and iterative formulae for sin(A/3) and sin(A/5).

The Siddhanta-tattva-viveka is a Sanskrit text and in it Kamalakara makes frequent use of the place-value number system with Sanskrit numerals. This and many other aspects of the work are discussed.

144579

Jagannatha Samrat

Born: about 1690 in Amber (now Jaipur), India

Died: about 1750 in India

Jagannatha had Jai Singh Sawai as his patron. Jai Singh Sawai was the ruler of Amber, now Jaipur, in eastern Rajasthan. He began his rule in 1699 and by clever use of tax rights on land that was rented by the state to an individual person he became the most important ruler in the region. His financial success let him finance the scholarly works of people such as Jagannatha. It is worth noting that Jai Singh's importance was recognised by Amber which was then called Jaipur in his honour.

Jai Singh ruled Amber throughout the period in which Jagannatha was producing his scientific work. He realised that the health of the country required Indian culture and science to be revitalised and returned to its position of leading importance which it had possessed. So Jai Singh employed Jagannatha to make Sanskrit translations of the important Greek scientific works which at that time were only available in Arabic translations. Jagannatha translated Euclid's Elements from the Arabic translation by Nasir al-Din al-Tusi made nearly 500 years earlier. His Sanskrit version

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was called Rekhaganita and it was completed by 1727. We know this date since a copy was made by a scribe and he dated the start of his work as 1727.

Ptolemy's Almagest had been one of the works which Arabic scientists had studied intently and, in 1247, al-Tusi wrote Tahrir al-Majisti (Commentary on the Almagest) in which he introduced various trigonometrical techniques to calculate tables of sines. Jagannatha translated al-Tusi's Arabic version but he did more than this for he included in the same work, which he called Siddhantasamrat, his own comments on related work of other Arabic mathematical astronomers such as Ulugh Beg and al-Kashi.

It is clear from Jagannatha's work that he is working as one of a group of mathematicians and astronomers gathered by Jai Singh in his scheme to bring the best in scientific ideas from outside India to reinvigorate the scientific scene in India.

Gupta looks at the history of the result

$\sin(\pi/10) = (O5 - 1)/4$

in Indian mathematics. The result appears for the first time in the work of Bhaskara II, but there were a number of interesting proofs of the result by later Indian mathematicians. One of the proofs presented by Gupta was by Jagannatha who gave a proof which was essentially geometric in nature but, interestingly, contained an analytic procedure in terms of trigonometric and algebraic steps.

Indian Numerals

It is worth beginning this article with the same quote from Laplace which we give in the article Overview of Indian mathematics. Laplace wrote:-

The ingenious method of expressing every possible number using a set of ten symbols (each symbol having a place value and an absolute value) emerged in

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India. The idea seems so simple nowadays that its significance and profound importance is no longer appreciated. Its simplicity lies in the way it facilitated calculation and placed arithmetic foremost amongst useful inventions, the importance of this invention is more readily appreciated when one considers that it was beyond the two greatest men of Antiquity, Archimedes and Apollonius.

The purpose of this article is to attempt the difficult task of trying to describe how the Indians developed this ingenious system. We will examine two different aspects of the Indian number systems in this article. First we will examine the way that the numerals 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 evolved into the form which we recognise today. Of course it is important to realise that there is still no standard way of writing these numerals. The different fonts on this computer can produce many forms of these numerals which, although recognisable, differ markedly from each other. Many hand-written versions are even hard to recognise.

The second aspect of the Indian number system which we want to investigate here is the place value system which, as Laplace comments in the quote which we gave at the beginning of this article, seems "so simple that its significance and profound importance is no longer appreciated." We should also note the fact, which is important to both aspects, that the Indian number systems are almost exclusively base 10, as opposed to the Babylonian base 60 systems.

Beginning with the numerals themselves, we certainly know that today's symbols took on forms close to that which they presently have in Europe in the 15th century. It was the advent of printing which motivated the standardisation of the symbols. However we must not forget that many countries use symbols today which are quite different from 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 and unless one learns these symbols they are totally unrecognisable as for example the Greek alphabet is to someone unfamiliar with it.

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One of the important sources of information which we have about Indian numerals comes from al-Biruni. During the 1020s al-Biruni made several visits to India. Before he went there al-Biruni already knew of Indian astronomy and mathematics from Arabic translations of some Sanskrit texts. In India he made a detailed study of Hindu philosophy and he also studied several branches of Indian science and mathematics.

Al-Biruni wrote 27 works on India and on different areas of the Indian sciences. In particular his account of Indian astronomy and mathematics is a valuable contribution to the study of the history of Indian science. Referring to the Indian numerals in a famous book written about 1030 he wrote:-

> *Whilst we use letters for calculation according to their numerical value, the Indians do not use letters at all for arithmetic. And just as the shape of the letters that they use for writing is different in different regions of their country, so the numerical symbols vary.*

It is reasonable to ask where the various symbols for numerals which al-Biruni saw originated. Historians trace them all back to the Brahmi numerals which came into being around the middle of the third century BC. Now these Brahmi numerals were not just symbols for the numbers between 1 and 9. The situation is much more complicated for it was not a place-value system so there were symbols for many more numbers. Also there were no special symbols for 2 and 3, both numbers being constructed from the symbol for 1.

1	2	3
—	=	≡

Brahmi one, two, three

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Here is the Brahmi one, two, three.

1	2	3	4	5	6	7	8	9
—	=	≡	+	h	६	7	৪	৯

Brahmi numerals around 1st century A.D.

There were separate Brahmi symbols for 4, 5, 6, 7, 8, 9 but there were also symbols for 10, 100, 1000, ... as well as 20, 30, 40, ... , 90 and 200, 300, 400, ..., 900.

The Brahmi numerals have been found in inscriptions in caves and on coins in regions near Poona, Bombay, and Uttar Pradesh. Dating these numerals tells us that they were in use over quite a long time span up to the 4th century AD. Of course different inscriptions differ somewhat in the style of the symbols.

Here is one style of the Brahmi numerals.

We should now look both forward and backward from the appearance of the Brahmi numerals. Moving forward leads to many different forms of numerals but we shall choose to examine only the path which has led to our present day symbols. First, however, we look at a number of different theories concerning the origin of the Brahmi numerals.

There is no problem in understanding the symbols for 1, 2, and 3. However the symbols for 4, ... , 9 appear to us to have no obvious link to the numbers they represent. There have been quite a number of theories put forward by historians over many years as to the origin of these numerals. Ifrah lists a number of the hypotheses which have been put forward.

1. The Brahmi numerals came from the Indus valley culture of around 2000 BC.
2. The Brahmi numerals came from Aramaean numerals.

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3. The Brahmi numerals came from the Karoshthi alphabet.
4. The Brahmi numerals came from the Brahmi alphabet.
5. The Brahmi numerals came from an earlier alphabetic numeral system, possibly due to Panini.
6. The Brahmi numerals came from Egypt.

1	2	3	4	5	6	7	8	9
I	II	III	II II	III II	III III	IIII III	IIII IIII	III III III

Ifrah's guess for pre-Brahmi numbers

1	2	3	4	5	6	7	8	9
—	=	≡	𑀔	𑀕	𑀖	𑀗	𑀘	𑀙

Gupta numerals around 4th century A.D.

Basically these hypotheses are of two types. One is that the numerals came from an alphabet in a similar way to the Greek numerals which were the initial letters of the names of the numbers. The second type of hypothesis is that they derive from an earlier number system of the same broad type as Roman numerals. For example the Aramaean numerals of hypothesis 2 are based on I (one) and X (four):

I, II, III, X, IX, IIX, IIIX, XX.

Ifrah examines each of the six hypotheses in turn and rejects them, although one would have to say that in some cases it is more due to lack of positive evidence rather than to negative evidence.

Ifrah proposes a theory of his own, namely that:-

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> *... the first nine Brahmi numerals constituted the vestiges of an old indigenous numerical notation, where the nine numerals were represented by the corresponding number of vertical lines ... To enable the numerals to be written rapidly, in order to save time, these groups of lines evolved in much the same manner as those of old Egyptian Pharonic numerals. Taking into account the kind of material that was written on in India over the centuries (tree bark or palm leaves) and the limitations of the tools used for writing (calamus or brush), the shape of the numerals became more and more complicated with the numerous ligatures, until the numerals no longer bore any resemblance to the original prototypes.*

It is a nice theory, and indeed could be true, but there seems to be absolutely no positive evidence in its favour. The idea is that they evolved from:

> *One might hope for evidence such as discovering numerals somewhere on this evolutionary path. However, it would appear that we will never find convincing proof for the origin of the Brahmi numerals.*

If we examine the route which led from the Brahmi numerals to our present symbols (and ignore the many other systems which evolved from the Brahmi numerals) then we next come to the Gupta symbols. The Gupta period is that during which the Gupta dynasty ruled over the Magadha state in northeastern India, and this was from the early 4th century AD to the late 6th century AD. The Gupta numerals developed from the Brahmi numerals and were spread over large areas by the Gupta empire as they conquered territory.

1	2	3	4	5	6	7	8	9	0
१	२	३	४	५	६	७	८	९	०

Nagari numerals around 11th century A.D.

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The Gupta numerals evolved into the Nagari numerals, sometimes called the Devanagari numerals. This form evolved from the Gupta numerals beginning around the 7th century AD and continued to develop from the 11th century onward. The name literally means the "writing of the gods" and it was the considered the most beautiful of all the forms which evolved. For example al-Biruni writes:-

What we [the Arabs] use for numerals is a selection of the best and most regular figures in India.

These "most regular figures" which al-Biruni refers to are the Nagari numerals which had, by his time, been transmitted into the Arab world. The way in which the Indian numerals were spread to the rest of the world between the 7th to the 16th centuries in examined in detail. In this paper, however, Gupta claims that Indian numerals had reached Southern Europe by the end of the 5th century but his argument is based on the Geometry of Boethius which is now known to be a forgery dating from the first half of the 11th century. It would appear extremely unlikely that the Indian numerals reach Europe as early as Gupta suggests.

We now turn to the second aspect of the Indian number system which we want to examine in this article, namely the fact that it was a place-value system with the numerals standing for different values depending on their position relative to the other numerals. Although our place-value system is a direct descendant of the Indian system, we should note straight away that the Indians were not the first to develop such a system. The Babylonians had a place-value system as early as the 19th century BC but the Babylonian systems were to base 60. The Indians were the first to develop a base 10 positional system and, considering the date of the Babylonian system, it came very late indeed.

The oldest dated Indian document which contains a number written in the place-value form used today is a legal document dated 346 in the Chhedi calendar which translates

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to a date in our calendar of 594 AD. This document is a donation charter of Dadda III of Sankheda in the Bharukachcha region. The only problem with it is that some historians claim that the date has been added as a later forgery. Although it was not unusual for such charters to be modified at a later date so that the property to which they referred could be claimed by someone who was not the rightful owner, there seems no conceivable reason to forge the date on this document. Therefore, despite the doubts, we can be fairly sure that this document provides evidence that a place-value system was in use in India by the end of the 6th century.

Many other charters have been found which are dated and use of the place-value system for either the date or some other numbers within the text. These include:

1. a donation charter of Dhiniki dated 794 in the Vikrama calendar which translates to a date in our calendar of 737 AD.
2. an inscription of Devendravarman dated 675 in the Shaka calendar which translates to a date in our calendar of 753 AD.
3. a donation charter of Danidurga dated 675 in the Shaka calendar which translates to a date in our calendar of 737 AD.
4. a donation charter of Shankaragana dated 715 in the Shaka calendar which translates to a date in our calendar of 793 AD.
5. a donation charter of Nagbhata dated 872 in the Vikrama calendar which translates to a date in our calendar of 815 AD.
6. an inscription of Bauka dated 894 in the Vikrama calendar which translates to a date in our calendar of 837 AD.

All of these are claimed to be forgeries by some historians but some, or all, may well be genuine.

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The first inscription which is dated and is not disputed is the inscription at Gwalior dated 933 in the Vikrama calendar which translates to a date in our calendar of 876 AD. Further details of this inscription is given in the article on zero.

There is indirect evidence that the Indians developed a positional number system as early as the first century AD. The evidence is found from inscriptions which, although not in India, have been found in countries which were assimilating Indian culture. Another source is the Bakhshali manuscript which contains numbers written in place-value notation. The problem here is the dating of this manuscript, a topic which is examined in detail in our article on the Bakhshali manuscript.

We are left, of course, with asking the question of why the Indians developed such an ingenious number system when the ancient Greeks, for example, did not. A number of theories have been put forward concerning this question. Some historians believe that the Babylonian base 60 place-value system was transmitted to the Indians via the Greeks. We have commented in the article on zero about Greek astronomers using the Babylonian base 60 place-value system with a symbol o similar to our zero. The theory here is that these ideas were transmitted to the Indians who then combined this with their own base 10 number systems which had existed in India for a very long time.

A second hypothesis is that the idea for place-value in Indian number systems came from the Chinese. In particular the Chinese had pseudo-positional number rods which, it is claimed by some, became the basis of the Indian positional system. This view is put forward by, for example, Lay Yong Lam. Lam argues that the Chinese system already contained what he calls the:-

> *... three essential features of our numeral notation system: (i) nine signs and the concept of zero, (ii) a place value system and (iii) a decimal base.*

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A third hypothesis is put forward by Joseph. His idea is that the place-value in Indian number systems is something which was developed entirely by the Indians. He has an interesting theory as to why the Indians might be pushed into such an idea. The reason, Joseph believes, is due to the Indian fascination with large numbers. Freudenthal is another historian of mathematics who supports the theory that the idea came entirely from within India.

To see clearly this early Indian fascination with large numbers, we can take a look at the Lalitavistara which is an account of the life of Gautama Buddha. It is hard to date this work since it underwent continuous development over a long period but dating it to around the first or second century AD is reasonable. In Lalitavistara Gautama, when he is a young man, is examined on mathematics. He is asked to name all the numerical ranks beyond a koti which is 107. He lists the powers of 10 up to 1053. Taking this as a first level he then carries on to a second level and gets eventually to 10421. Gautama's examiner says:-

You, not I, are the master mathematician.

It is stories such as this, and many similar ones, which convince Joseph that the fascination of the Indians with large numbers must have driven them to invent a system in which such numbers are easily expressed, namely a place-valued notation. He writes:-

> *The early use of such large numbers eventually led to the adoption of a series of names for successive powers of ten. The importance of these number names cannot be exaggerated. The word-numeral system, later replaced by an alphabetic notation, was the logical outcome of proceeding by multiples of ten. ... The decimal place-value system developed when a decimal scale came to be associated with the value of the places of the numbers arranged left to right or right to left and this was precisely what happened in India ...*

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However, the same story in Lalitavistara convinces Kaplan that the Indians' ideas of numbers came from the Greeks, for to him the story is an Indian version of Archimedes' Sand-reckoner. All that we know is that the place-value system of the Indians, however it arose, was transmitted to the Arabs and later into Europe to have, in the words of Laplace, profound importance on the development of mathematics.

Jaina Mathematics

It is a little hard to define Jaina mathematics. Jainism is a religion and philosophy which was founded in India around the 6th century BC. To a certain extent it began to replace the Vedic religions which, with their sacrificial procedures, had given rise to the mathematics of building altars. The mathematics of the Vedic religions is described in the article Indian Sulbasutras.

Now we could use the term Jaina mathematics to describe mathematics done by those following Jainism and indeed this would then refer to a part of mathematics done on the Indian subcontinent from the founding of Jainism up to modern times. Indeed this is fair and some of the articles in the references refer to fairly modern mathematics. For example, Jha looks at the contributions of Jainas from the 5th century BC up to the 18th century AD.

This article will concentrate on the period after the founding of Jainism up to around the time of Aryabhatta in around 500 AD. The reason for taking this time interval is that until recently this was thought to be a time when there was little mathematical activity in India. Aryabhatta's work was seen as the beginning of a new classical period for Indian mathematics and indeed this is fair. Yet Aryabhatta did not work in mathematical isolation and as well as being seen as the person who brought in a new era of mathematical investigation in India, more recent research has shown that

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there is a case for seeing him also as representing the end-product of a mathematical period of which relatively little is known. This is the period we shall refer to as the period of Jaina mathematics.

There were mathematical texts from this period yet they have received little attention from historians until recent times. Texts, such as the Surya Prajnapti which is thought to be around the 4th century BC and the Jambudvipa Prajnapti from around the same period, have recently received attention through the study of later commentaries. The Bhagabati Sutra dates from around 300 BC and contains interesting information on combinations. From about the second century BC is the Sthananga Sutra which is particularly interesting in that it lists the topics which made up the mathematics studied at the time. In fact this list of topics sets the scene for the areas of study for a long time to come in the Indian subcontinent. The topics are listed as:-

> *... the theory of numbers, arithmetical operations, geometry, operations with fractions, simple equations, cubic equations, quartic equations, and permutations and combinations.*

The ideas of the mathematical infinite in Jaina mathematics is very interesting indeed and they evolve largely due to the Jaina's cosmological ideas. In Jaina cosmology time is thought of as eternal and without form. The world is infinite, it was never created and has always existed. Space pervades everything and is without form. All the objects of the universe exist in space which is divided into the space of the universe and the space of the non-universe. There is a central region of the universe in which all living beings, including men, animals, gods and devils, live. Above this central region is the upper world which is itself divided into two parts. Below the central region is the lower world which is divided into seven tiers. This led to the work described on a mathematical topic in the Jaina work,

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Tiloyapannatti by Yativrsabha. A circle is divided by parallel lines into regions of prescribed widths. The lengths of the boundary chords and the areas of the regions are given, based on stated rules.

This cosmology has strongly influenced Jaina mathematics in many ways and has been a motivating factor in the development of mathematical ideas of the infinite which were not considered again until the time of Cantor. The Jaina cosmology contained a time period of 2^{588} years. Note that 2^{588} is a very large number!

2^{588} = 1013 065324 433836 171511 818326 096474 890383 898005 918563 696288 002277 756507 034036 354527 929615 978746 851512 277392 062160 962106 733983 191180 520452 956027 069051 297354 415786 421338 721071 661056.

So what are the Jaina ideas of the infinite. There was a fascination with large numbers in Indian thought over a long period and this again almost required them to consider infinitely large measures. The first point worth making is that they had different infinite measures which they did not define in a rigorous mathematical fashion, but nevertheless are quite sophisticated. The paper describes the way that the first unenumerable number was constructed using effectively a recursive construction.

The Jaina construction begins with a cylindrical container of very large radius r^q (taken to be the radius of the earth) and having a fixed height h. The number $n^q = f(r^q)$ is the number of very tiny white mustard seeds that can be placed in this container. Next, $r_1 = g(r^q)$ is defined by a complicated recursive subprocedure, and then as before a new larger number $n_1 = f(r_1)$ is defined. The text the Anuyoga Dwara Sutra then states:-

Still the highest enumerable number has not been attained.

The whole procedure is repeated, yielding a truly huge number which is called jaghanya- parita- asamkhyata

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meaning "unenumerable of low enhanced order". Continuing the process yields the smallest unenumerable number.

Jaina mathematics recognised five different types of infinity:-

> *... infinite in one direction, infinite in two directions, infinite in area, infinite everywhere and perpetually infinite.*

The Anuyoga Dwara Sutra contains other remarkable numerical speculations by the Jainas. For example several times in the work the number of human beings that ever existed is given as 2^{96}.

By the second century AD the Jaina had produced a theory of sets. In Satkhandagama various sets are operated upon by logarithmic functions to base two, by squaring and extracting square roots, and by raising to finite or infinite powers. The operations are repeated to produce new sets.

Permutations and combinations are used in the Sthananga Sutra. In the Bhagabati Sutra rules are given for the number of permutations of 1 selected from n, 2 from n, and 3 from n. Similarly rules are given for the number of combinations of 1 from n, 2 from n, and 3 from n. Numbers are calculated in the cases where n = 2, 3 and 4. The author then says that one can compute the numbers in the same way for larger n. He writes:-

> *In this way, 5, 6, 7, ..., 10, etc. or an enumerable, unenumerable or infinite number of may be specified. Taking one at a time, two at a time, ... ten at a time, as the number of combinations are formed they must all be worked out.*

Interestingly here too there is the suggestion that the arithmetic can be extended to various infinite numbers. In other works the relation of the number of combinations to the coefficients occurring in the binomial expansion was noted. In a commentary on this third century work in the

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tenth century, Pascal's triangle appears in order to give the coefficients of the binomial expansion.

Another concept which the Jainas seem to have gone at least some way towards understanding was that of the logarithm. They had begun to understand the laws of indices. For example the Anuyoga Dwara Sutra states:-

The first square root multiplied by the second square root is the cube of the second square root.

The second square root was the fourth root of a number. This therefore is the formula

$(Oa).(OOa) = (OOa)^3$.

Again the Anuyoga Dwara Sutra states:-

... the second square root multiplied by the third square root is the cube of the third square root.

The third square root was the eighth root of a number. This therefore is the formula

$(OOa).(OOOa) = (OOOa)^3$.

Some historians studying these works believe that they see evidence for the Jainas having developed logarithms to base 2.

The value of p in Jaina mathematics has been a topic of a number of research papers. As with much research into Indian mathematics there is interest in whether the Indians took their ideas from the Greeks. The approximation $\pi = O10$ seems one which was frequently used by the Jainas.

Finally let us comment on the Jaina's astronomy. This was not very advanced. It was not until the works of Aryabhatta that the Greek ideas of epicycles entered Indian astronomy. Before the Jaina period the ideas of eclipses were based on a demon called Rahu which devoured or captured the Moon or the Sun causing their eclipse. The Jaina school assumed the existence of two demons Rahu, the

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Dhruva Rahu which causes the phases of the Moon and the Parva Rahu which has irregular celestial motion in all directions and causes an eclipse by covering the Moon or Sun or their light. The author of points out that, according to the Jaina school, the greatest possible number of eclipses in a year is four.

Despite this some of the astronomical measurements were fairly good. The data in the Surya Prajnapti implies a synodic lunar month equal to 29 plus $\frac{16}{31}$ days; the correct value being nearly 29.5305888. There has been considerable interest in examining the data presented in these Jaina texts to see if the data originated from other sources. For example in the Surya Prajnapti data exists which implies a ratio of 3:2 for the maximum to the minimum length of daylight. Now this is not true for India but is true for Babylonia which makes some historians believe that the data in the Surya Prajnapti is not of Indian origin but is Babylonian. However, Sharma and Lishk present an alternative hypothesis which would allow the data to be of Indian origin. One has to say that their suggestion that 3:2 might be the ratio of the amounts of water to be poured into the water-clock on the longest and shortest days seems less than totally convincing.

The Bakhshali Manuscript

The Bakhshali manuscript is an early mathematical manuscript which was discovered over 100 years ago. We shall discuss in a moment the problem of dating this manuscript, a topic which has aroused much controversy, but for the moment we will examine how it was discovered. The paper describes this discovery along with the early history of the manuscript. Gupta writes:-

The Bakhshali Manuscript is the name given to the mathematical work written on birch bark and found

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in the summer of 1881 near the village Bakhshali (or Bakhshalai) of the Yusufzai subdivision of the Peshawar district (now in Pakistan). The village is in Mardan tahsil and is situated 50 miles from the city of Peshawar.

An Inspector of Police named Mian An-Wan-Udin (whose tenant actually discovered the manuscript while digging a stone enclosure in a ruined place) took the work to the Assistant Commissioner at Mardan who intended to forward the manuscript to Lahore Museum. However, it was subsequently sent to the Lieutenant Governor of Punjab who, on the advice of General A Cunningham, directed it to be passed on to Dr. Rudolf Hoernle of the Calcutta Madrasa for study and publication. Dr. Hoernle presented a description of the BM before the Asiatic Society of Bengal in 1882, and this was published in the Indian Antiquary in 1883. He gave a fuller account at the Seventh Oriental Conference held at Vienna in 1886 and this was published in its Proceedings. A revised version of this paper appeared in the Indian Antiquary of 1888. In 1902, he presented the Bakhshali Manuscript to the Bodleian Library, Oxford, where it is still (Shelf mark: MS. Sansk. d. 14).

A large part of the manuscript had been destroyed and only about 70 leaves of birch-bark, of which a few were only scraps, survived to the time of its discovery.

To show the arguments regarding its age we note that F R Hoernle, referred to in the quotation above, placed the Bakhshali manuscript between the third and fourth centuries AD. Many other historians of mathematics such as Moritz Cantor, F Cajori, B Datta, S N Sen, A K Bag, and R C Gupta agreed with this dating. In 1927-1933 the Bakhshali manuscript was edited by G R Kaye and published with a comprehensive introduction, an English translation, and a transliteration together with facsimiles of the text. Kaye claimed that the manuscript dated from the twelfth century AD and he even doubted that it was of Indian origin.

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Channabasappa gives the range 200 - 400 AD as the most probable date. The same author identifies five specific mathematical terms which do not occur in the works of Aryabhatta and he argues that this strongly supports a date for the Bakhshali manuscript earlier than the 5th century. Joseph suggests that the evidence all points to the:-

> *... manuscript [being] probably a later copy of a document composed at some time in the first few centuries of the Christian era.*

L V Gurjar claims that the manuscript is no later than 300 AD. On the other hand T Hayashi claims that the date of the original is probably from the seventh century, but he also claims that the manuscript itself is a later copy which was made between the eighth and the twelfth centuries AD.

I [EFR] feel that if one weighs all the evidence of these experts the most likely conclusion is that the manuscript is a later copy of a work first composed around 400 AD. Why do I believe that the actual manuscript was written later? Well our current understanding of Indian numerals and writing would date the numerals used in the manuscript as not having appeared before the ninth or tenth century. To accept that this style of numeral existed in 400 AD. would force us to change greatly our whole concept of the time-scale for the development of Indian numerals. Sometimes, of course, we are forced into major rethinks but, without supporting evidence, everything points to the manuscript being a tenth century copy of an original from around 400 AD. Despite the claims of Kaye, it is essentially certain that the manuscript is Indian.

The attraction of the date of 400 AD for the Bakhshali manuscript is that this puts it just before the "classical period" of Indian mathematics which began with the work of Aryabhatta I around 500. It would then fill in knowledge we have of Indian mathematics for, prior to the discovery of this manuscript, we had little knowledge of Indian

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mathematics between the dates of about 200 BC, and 500 AD. This date would make it a document near the end of the period of Jaina mathematics and it can be seen as, in some sense, marking the achievements of the Jains.

What does the manuscript contain? Joseph writes:-

The Bakhshali manuscript is a handbook of rules and illustrative examples together with their solutions. It is devoted mainly to arithmetic and algebra, with just a few problems on geometry and mensuration. Only parts of it have been restored, so we cannot be certain about the balance between different topics.

3 1+ 4 2	means 3/4 minus 1/2

1 1 3	means 1 plus 1/3 so equals 4/3

Now the way that the manuscript is laid out is quite unusual for an Indian document (which of course leads people like Kaye to prefer the hypothesis that it is not Indian at all - an idea in which we cannot see any merit). The Bakhshali manuscript gives the statement of a rule. There then follows an example given first in words, then using mathematical notation. The solution to the example is then given and finally a proof is set out.

The notation used is not unlike that used by Aryabhatta but it does have features not found in any other document. Fractions are not dissimilar in notation to that used today, written with one number below the other. No line appears between the numbers as we would write today, however. Another unusual feature is the sign + placed after a number to indicate a negative. It is very strange for us today to see

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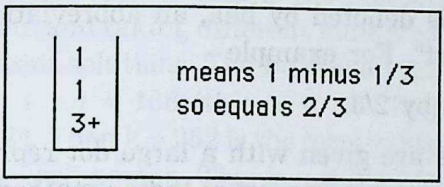
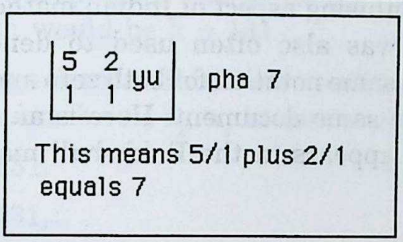
our addition symbol being used for subtraction. As an example, here is how $\frac{3}{4}-\frac{1}{2}$ would be written.

$\frac{3}{4}$ min us $\frac{1}{2}$





Compound fractions were written in three lines. Hence 1 plus 1/3 would be written thus

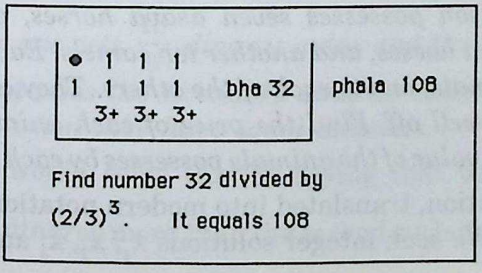
1 plus 1/3

and 1 minus 1/3 = 2/3 in the following way

1 minus 1/3 = 2/3

Sums of fractions such as 5/1 plus 2/1 are written using the symbol yu (for yuta)

5/1 plus 2/1



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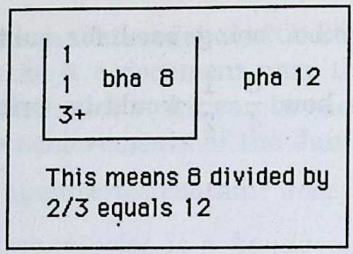

Division is denoted by bha, an abbreviation for bhaga meaning "part". For example

8 divided by 2/3

Equations are given with a large dot representing the unknown. A confusing aspect of Indian mathematics is that this notation was also often used to denote zero, and sometimes this same notation for both zero and the unknown are used in the same document. Here is an example of an equation as it appears in the Bakhshali manuscript.

EQUATION

The method of equalisation is found in many types of problems which occur in the manuscript. Problems of this type which are found in the manuscript are examined and some of these lead to indeterminate equations. Included are problems concerning equalising wealth, the positions of two travellers, wages, and purchases by a number of merchants. These problems can all be reduced to solving a linear equation with one unknown or to a system of n linear equations in n unknowns. To illustrate we give the following indeterminate problem which, of course, does not have a unique solution:-

> *One person possesses seven asava horses, another nine haya horses, and another ten camels. Each gives two animals, one to each of the others. They are then equally well off. Find the price of each animal and the total value of the animals possesses by each person.*

The solution, translated into modern notation, proceeds as follows. We seek integer solutions x_1, x_2, x_3 and k (where

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x_1 is the price of an asava, x_2 is the price of a haya, and x_3 is the price of a horse) satisfying

$$5x_1 + x_2 + x_3 = x_1 + 7x_2 + x_3 = x_1 + x_2 + 8x_3 = k.$$

Then $4x_1 = 6x_2 = 7x_3 = k - (x_1 + x_2 + x_3)$.

For integer solutions $k - (x_1 + x_2 + x_3)$ must be a multiple of the lcm of 4, 6 and 7. This is the indeterminate nature of the problem and taking different multiples of the lcm will lead to different solutions. The Bakhshali manuscript takes $k - (x_1 + x_2 + x_3) = 168$ (this is 4 6 7) giving $x_1 = 42$, $x_2 = 28$, $x_3 = 24$. Then $k = 262$ is the total value of the animals possesses by each person. This is not the minimum integer solution which would be $k = 131$.

If we use modern methods we would solve the system of three equation for x_1, x_2, x_3 in terms of k to obtain

$$x_1 = 21k/131,$$

$$x_2 = 14k/131,$$

$$x_3 = 12k/131$$

so we obtain integer solutions by taking $k = 131$ which is the smallest solution. This solution is not given in the Bakhshali manuscript but the author of the manuscript would have obtained this had he taken $k - (x_1 + x_2 + x_3) = \text{lcm}(4, 6, 7) = 84$.

Here is another equalisation problem taken from the manuscript which has a unique solution:-

Two page-boys are attendants of a king. For their services one gets $\frac{13}{6}$ *dinaras a day and the other* $\frac{3}{2}$. *The first owes the second 10 dinaras. calculate and tell me when they have equal amounts.*

Now I would solve this by saying that the first gets $\frac{13}{6} - \frac{3}{2} = \frac{2}{3}$ dinaras more than the second each day. He needs

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20 dinaras more than the second to be able to give back his 10 dinaras debt and have them with equal amounts. So 30 days are required when each has $13 \frac{30}{6} - 10 = 55$ dinaras. This is not the method of the Bakhshali manuscript which uses the "rule of three".

The rule of three is the familiar way of solving problems of the type: if a man earns 50 dinaras in 8 days how much will he earn in 12 days. The Bakhshali manuscript describes the rule where the three numbers are written

8 50 12

The 8 is the "pramana", the 50 is the "phala" and the 12 is the "iccha". The rule, according to the Bakhshali manuscript gives the answer as

phala × iccha/pramana

or in the case of the example $50 \times \frac{12}{8} = 75$ dinaras.

Applying this to the page-boy problem we obtain equal amounts for the page-boys after n days where

$$13 \times \frac{n}{6} = 3 \times \frac{n}{2} + 20$$

so $n = 30$ and each has $13 \times \frac{30}{6} - 10 = 55$ dinaras.

Another interesting piece of mathematics in the manuscript concerns calculating square roots. The following formula is used

$$OQ = O(A^2 + b) = A + b/2A - (b/2A)2/(2(A + b/2A))$$

This is stated in the manuscript as follows:-

In the case of a non-square number, subtract the nearest square number, divide the remainder by twice this nearest square; half the square of this is divided

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by the sum of the approximate root and the fraction this is subtracted and will give the corrected root.

Taking Q = 41, then A = 6, b = 5 and we obtain 6.403138528 as the approximation to O41 = 6.403124237. Hence we see that the Bakhshali formula gives the result correct to four decimal places.

The Bakhshali manuscript also uses the formula to compute O105 giving 10.24695122 as the approximation to O105 ≐ 10.24695077. This time the Bakhshali formula gives the result correct to five decimal places.

The following examples also occur in the Bakhshali manuscript where the author applies the formula to obtain approximate square roots:

O487

Bakhshali formula gives 22.068076490965

Correct answer is 22.068076490713

Here 9 decimal places are correct

O889

Bakhshali formula gives 29.816105242176

Correct answer is 29.8161030317511

Here 5 decimal places are correct

[Note. If we took 889 = 30^2 - 11 instead of 29^2 + 48 we would get

Bakhshali formula gives 29.816103037078

Correct answer is 29.8161030317511

Here 8 decimal places are correct]

O339009

Bakhshali formula gives 582.2447938796899

Correct answer is 582.2447938796876

Here 11 decimal places are correct

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It is interesting to note that Channabasappa derives from the Bakhshali square root formula an iterative scheme for approximating square roots. He finds that it is 38% faster than Newton's method in giving v41 to ten places of decimals.

A History of Zero

One of the commonest questions which the readers of this archive ask is: Who discovered zero? Why then have we not written an article on zero as one of the first in the archive? The reason is basically because of the difficulty of answering the question in a satisfactory form. If someone had come up with the concept of zero which everyone then saw as a brilliant innovation to enter mathematics from that time on, the question would have a satisfactory answer even if we did not know which genius invented it. The historical record, however, shows quite a different path towards the concept. Zero makes shadowy appearances only to vanish again almost as if mathematicians were searching for it yet did not recognise its fundamental significance even when they saw it.

The first thing to say about zero is that there are two uses of zero which are both extremely important but are somewhat different. One use is as an empty place indicator in our place-value number system. Hence in a number like 2106 the zero is used so that the positions of the 2 and 1 are correct. Clearly 216 means something quite different. The second use of zero is as a number itself in the form we use it as 0. There are also different aspects of zero within these two uses, namely the concept, the notation, and the name. (Our name "zero" derives ultimately from the Arabic sifr which also gives us the word "cipher".)

Neither of the above uses has an easily described history. It just did not happen that someone invented the ideas, and then everyone started to use them. Also it is fair to say that

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the number zero is far from an intuitive concept. Mathematical problems started as 'real' problems rather than abstract problems. Numbers in early historical times were thought of much more concretely than the abstract concepts which are our numbers today. There are giant mental leaps from 5 horses to 5 "things" and then to the abstract idea of "five". If ancient peoples solved a problem about how many horses a farmer needed then the problem was not going to have 0 or -23 as an answer.

One might think that once a place-value number system came into existence then the 0 as an empty place indicator is a necessary idea, yet the Babylonians had a place-value number system without this feature for over 1000 years. Moreover there is absolutely no evidence that the Babylonians felt that there was any problem with the ambiguity which existed. Remarkably, original texts survive from the era of Babylonian mathematics. The Babylonians wrote on tablets of unbaked clay, using cuneiform writing. The symbols were pressed into soft clay tablets with the slanted edge of a stylus and so had a wedge-shaped appearance (and hence the name cuneiform). Many tablets from around 1700 BC survive and we can read the original texts. Of course their notation for numbers was quite different from ours (and not based on 10 but on 60) but to translate into our notation they would not distinguish between 2106 and 216 (the context would have to show which was intended). It was not until around 400 BC that the Babylonians put two wedge symbols into the place where we would put zero to indicate which was meant, 216 or 21 " 6.

The two wedges were not the only notation used, however, and on a tablet found at Kish, an ancient Mesopotamian city located east of Babylon in what is today south-central Iraq, a different notation is used. This tablet, thought to date from around 700 BC, uses three hooks to denote an empty place in the positional notation. Other tablets dated from around the same time use a single hook for an empty place. There

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is one common feature to this use of different marks to denote an empty position. This is the fact that it never occurred at the end of the digits but always between two digits. So although we find 21 " 6 we never find 216 ". One has to assume that the older feeling that the context was sufficient to indicate which was intended still applied in these cases.

If this reference to context appears silly then it is worth noting that we still use context to interpret numbers today. If I take a bus to a nearby town and ask what the fare is then I know that the answer "It's three fifty" means three pounds fifty pence. Yet if the same answer is given to the question about the cost of a flight from Edinburgh to New York then I know that three hundred and fifty pounds is what is intended.

We can see from this that the early use of zero to denote an empty place is not really the use of zero as a number at all, merely the use of some type of punctuation mark so that the numbers had the correct interpretation.

Now the ancient Greeks began their contributions to mathematics around the time that zero as an empty place indicator was coming into use in Babylonian mathematics. The Greeks however did not adopt a positional number system. It is worth thinking just how significant this fact is. How could the brilliant mathematical advances of the Greeks not see them adopt a number system with all the advantages that the Babylonian place-value system possessed? The real answer to this question is more subtle than the simple answer that we are about to give, but basically the Greek mathematical achievements were based on geometry. Although Euclid's Elements contains a book on number theory, it is based on geometry. In other words Greek mathematicians did not need to name their numbers since they worked with numbers as lengths of lines. Numbers which required to be named for records were used by

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merchants, not mathematicians, and hence no clever notation was needed.

Now there were exceptions to what we have just stated. The exceptions were the mathematicians who were involved in recording astronomical data. Here we find the first use of the symbol which we recognise today as the notation for zero, for Greek astronomers began to use the symbol O. There are many theories why this particular notation was used. Some historians favour the explanation that it is omicron, the first letter of the Greek word for nothing namely "ouden". Neugebauer, however, dismisses this explanation since the Greeks already used omicron as a number - it represented 70 (the Greek number system was based on their alphabet). Other explanations offered include the fact that it stands for "obol", a coin of almost no value, and that it arises when counters were used for counting on a sand board. The suggestion here is that when a counter was removed to leave an empty column it left a depression in the sand which looked like O.

Ptolemy in the Almagest written around 130 AD uses the Babylonian sexagesimal system together with the empty place holder O. By this time Ptolemy is using the symbol both between digits and at the end of a number and one might be tempted to believe that at least zero as an empty place holder had firmly arrived. This, however, is far from what happened. Only a few exceptional astronomers used the notation and it would fall out of use several more times before finally establishing itself. The idea of the zero place (certainly not thought of as a number by Ptolemy who still considered it as a sort of punctuation mark) makes its next appearance in Indian mathematics.

The scene now moves to India where it is fair to say the numerals and number system was born which have evolved into the highly sophisticated ones we use today. Of course that is not to say that the Indian system did not owe

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something to earlier systems and many historians of mathematics believe that the Indian use of zero evolved from its use by Greek astronomers. As well as some historians who seem to want to play down the contribution of the Indians in a most unreasonable way, there are also those who make claims about the Indian invention of zero which seem to go far too far. For example Mukherjee claims:-

> *... the mathematical conception of zero ... was also present in the spiritual form from 17 000 years back in India.*

What is certain is that by around 650 AD the use of zero as a number came into Indian mathematics. The Indians also used a place-value system and zero was used to denote an empty place. In fact there is evidence of an empty place holder in positional numbers from as early as 200 AD in India but some historians dismiss these as later forgeries. Let us examine this latter use first since it continues the development described above.

In around 500 AD Aryabhatta devised a number system which has no zero yet was a positional system. He used the word "kha" for position and it would be used later as the name for zero. There is evidence that a dot had been used in earlier Indian manuscripts to denote an empty place in positional notation. It is interesting that the same documents sometimes also used a dot to denote an unknown where we might use x. Later Indian mathematicians had names for zero in positional numbers yet had no symbol for it. The first record of the Indian use of zero which is dated and agreed by all to be genuine was written in 876.

We have an inscription on a stone tablet which contains a date which translates to 876. The inscription concerns the town of Gwalior, 400 km south of Delhi, where they planted a garden 187 by 270 hastas which would produce enough flowers to allow 50 garlands per day to be given to the local temple. Both of the numbers 270 and 50 are denoted almost

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as they appear today although the 0 is smaller and slightly raised.

We now come to considering the first appearance of zero as a number. Let us first note that it is not in any sense a natural candidate for a number. From early times numbers are words which refer to collections of objects. Certainly the idea of number became more and more abstract and this abstraction then makes possible the consideration of zero and negative numbers which do not arise as properties of collections of objects. Of course the problem which arises when one tries to consider zero and negatives as numbers is how they interact in regard to the operations of arithmetic, addition, subtraction, multiplication and division. In three important books the Indian mathematicians Brahmagupta, Mahavira and Bhaskara tried to answer these questions.

Brahmagupta attempted to give the rules for arithmetic involving zero and negative numbers in the seventh century. He explained that given a number then if you subtract it from itself you obtain zero. He gave the following rules for addition which involve zero:-

The sum of zero and a negative number is negative, the sum of a positive number and zero is positive, the sum of zero and zero is zero.

Subtraction is a little harder:-

A negative number subtracted from zero is positive, a positive number subtracted from zero is negative, zero subtracted from a negative number is negative, zero subtracted from a positive number is positive, zero subtracted from zero is zero.

Brahmagupta then says that any number when multiplied by zero is zero but struggles when it comes to division:-

A positive or negative number when divided by zero is a fraction with the zero as denominator. Zero divided by a

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negative or positive number is either zero or is expressed as a fraction with zero as numerator and the finite quantity as denominator. Zero divided by zero is zero.

Really Brahmagupta is saying very little when he suggests that n divided by zero is n/0. Clearly he is struggling here. He is certainly wrong when he then claims that zero divided by zero is zero. However it is a brilliant attempt from the first person that we know who tried to extend arithmetic to negative numbers and zero.

In 830, around 200 years after Brahmagupta wrote his masterpiece, Mahavira wrote Ganita Sara Samgraha which was designed as an updating of Brahmagupta's book. He correctly states that:-

> *... a number multiplied by zero is zero, and a number remains the same when zero is subtracted from it.*

However his attempts to improve on Brahmagupta's statements on dividing by zero seem to lead him into error. He writes:-

> A number remains unchanged when divided by zero.

Since this is clearly incorrect my use of the words "seem to lead him into error" might be seen as confusing. The reason for this phrase is that some commentators on Mahavira have tried to find excuses for his incorrect statement. Bhaskara wrote over 500 years after Brahmagupta. Despite the passage of time he is still struggling to explain division by zero. He writes:-

> *A quantity divided by zero becomes a fraction the denominator of which is zero. This fraction is termed an infinite quantity. In this quantity consisting of that which has zero for its divisor, there is no alteration, though many may be inserted or extracted; as no change takes place in the infinite and immutable God when worlds are created or destroyed, though numerous orders of beings are absorbed or put forth.*

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So Bhaskara tried to solve the problem by writing n/0 = 8. At first sight we might be tempted to believe that Bhaskara has it correct, but of course he does not. If this were true then 0 times 8 must be equal to every number n, so all numbers are equal. The Indian mathematicians could not bring themselves to the point of admitting that one could not divide by zero. Bhaskara did correctly state other properties of zero, however, such as $0^2 = 0$, and O0 = 0.

Perhaps we should note at this point that there was another civilisation which developed a place-value number system with a zero. This was the Maya people who lived in central America, occupying the area which today is southern Mexico, Guatemala, and northern Belize. This was an old civilisation but flourished particularly between 250 and 900. We know that by 665 they used a place-value number system to base 20 with a symbol for zero. However their use of zero goes back further than this and was in use before they introduced the place-valued number system. This is a remarkable achievement but sadly did not influence other peoples.

You can see a separate article about Mayan mathematics.

The brilliant work of the Indian mathematicians was transmitted to the Islamic and Arabic mathematicians further west. It came at an early stage for al-Khwarizmi wrote Al'Khwarizmi on the Hindu Art of Reckoning which describes the Indian place-value system of numerals based on 1, 2, 3, 4, 5, 6, 7, 8, 9, and 0. This work was the first in what is now Iraq to use zero as a place holder in positional base notation. Ibn Ezra, in the 12th century, wrote three treatises on numbers which helped to bring the Indian symbols and ideas of decimal fractions to the attention of some of the learned people in Europe. The Book of the Number describes the decimal system for integers with place values from left to right. In this work ibn Ezra uses zero which he calls galgal (meaning wheel or circle). Slightly later in the 12th century al-Samawal was writing:-

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If we subtract a positive number from zero the same negative number remains. ... if we subtract a negative number from zero the same positive number remains.

The Indian ideas spread east to China as well as west to the Islamic countries. In 1247 the Chinese mathematician Ch'in Chiu-Shao wrote Mathematical treatise in nine sections which uses the symbol O for zero. A little later, in 1303, Zhu Shijie wrote Jade mirror of the four elements which again uses the symbol O for zero. Fibonacci was one of the main people to bring these new ideas about the number system to Europe. As the authors write:-

An important link between the Hindu-Arabic number system and the European mathematics is the Italian mathematician Fibonacci.

In Liber Abaci he described the nine Indian symbols together with the sign 0 for Europeans in around 1200 but it was not widely used for a long time after that. It is significant that Fibonacci is not bold enough to treat 0 in the same way as the other numbers 1, 2, 3, 4, 5, 6, 7, 8, 9 since he speaks of the "sign" zero while the other symbols he speaks of as numbers. Although clearly bringing the Indian numerals to Europe was of major importance we can see that in his treatment of zero he did not reach the sophistication of the Indians Brahmagupta, Mahavira and Bhaskara nor of the Arabic and Islamic mathematicians such as al-Samawal.

One might have thought that the progress of the number systems in general, and zero in particular, would have been steady from this time on. However, this was far from the case. Cardan solved cubic and quartic equations without using zero. He would have found his work in the 1500's so much easier if he had a zero but it was not part of his mathematics. By the 1600's zero began to come into widespread use but still only after encountering a lot of resistance.

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Of course there are still signs of the problems caused by zero. Recently many people throughout the world celebrated the new millennium on 1 January 2000. Of course they celebrated the passing of only 1999 years since when the calendar was set up no year zero was specified. Although one might forgive the original error, it is a little surprising that most people seemed unable to understand why the third millennium and the 21st century begin on 1 January 2001. Zero is still causing problems!

A History of Pi

A little known verse of the Bible reads:

And he made a molten sea, ten cubits from the one brim to the other: it was round all about, and his height was five cubits: and a line of thirty cubits did compass it about. (I Kings 7, 23)

The same verse can be found in II Chronicles 4, 2. It occurs in a list of specifications for the great temple of Solomon, built around 950 BC and its interest here is that it gives $\pi = 3$. Not a very accurate value of course and not even very accurate in its day, for the Egyptian and Mesopotamian values of $\frac{25}{8} = 3.125$ and O10 = 3.162 have been traced to much earlier dates: though in defence of Solomon's craftsmen it should be noted that the item being described seems to have been a very large brass casting, where a high degree of geometrical precision is neither possible nor necessary. There are some interpretations of this which lead to a much better value.

The fact that the ratio of the circumference to the diameter of a circle is constant has been known for so long that it is quite untraceable. The earliest values of p including the 'Biblical' value of 3, were almost certainly found by measurement. In the Egyptian Rhind Papyrus, which is

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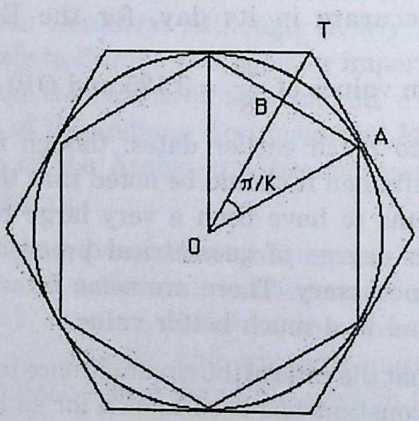
dated about 1650 BC, there is good evidence for $4 \times (8/9)^2 = 3.16$ as a value for π. The first theoretical calculation seems to have been carried out by Archimedes of Syracuse (287-212 BC). He obtained the approximation

$$\frac{223}{71} < \pi < \frac{22}{7}.$$

Before giving an indication of his proof, notice that very considerable sophistication involved in the use of inequalities here. Archimedes knew, what so many people to this day do not, that p does not equal $\frac{22}{7}$, and made no claim to have discovered the exact value. If we take his best estimate as the average of his two bounds we obtain 3.1418, an error of about 0.0002.

Here is Archimedes' argument.

Consider a circle of radius 1, in which we inscribe a regular polygon of $3 \times 2n^{-1}$ sides, with semiperimeter bn, and superscribe a regular polygon of $3 \times 2n^{-1}$ sides, with semiperimeter an.



OA = 1
AB = sin(π/K)
AT = tan(π/K)
where $K = 3 \times 2^{n-1}$

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The diagram for the case n = 2 is on the right.

The effect of this procedure is to define an increasing sequence b_1 , b_2 , b_3 , ...

and a decreasing sequence a_1 , a_2 , a_3 , ...

such that both sequences have limit p.

Using trigonometrical notation, we see that the two semiperimeters are given by

$a_n = K \tan(\pi/K)$, bn = $K \sin(\pi/K)$,

where $K = 3 \times 2n^{-1}$. Equally, we have

$a_{n+1} = 2K \tan(\pi/2K)$, $b_{n+1} = 2K \sin(\pi/2K)$,

and it is not a difficult exercise in trigonometry to show that.

$$\left(\frac{1}{a_n}+\frac{1}{b_n}\right)=\frac{2}{a_{n+1}} \quad ...(1)$$

$$a_{n+1}b_n = \left(b_{n+1}\right)^2 \quad ...(2)$$

Archimedes, starting from $a_1 = 3 \tan(\pi/3) = 3O3$ and $b_1 = 3 \sin(\pi/3) = 3O3/2$, calculated a_2 using (1), then b_2 using (2), then a_3 using (1), then b_3 using (2), and so on until he had calculated a_6 and b_6. His conclusion was that

$b_6 < \pi < a_6$.

It is important to realise that the use of trigonometry here is unhistorical: Archimedes did not have the advantage of an algebraic and trigonometrical notation and had to derive (1) and (2) by purely geometrical means. Moreover he did not even have the advantage of our decimal notation for numbers, so that the calculation of a6 and b6 from (1) and (2) was by no means a trivial task. So it was a pretty stupendous feat both of imagination and of calculation and the wonder is not that he stopped with polygons of 96 sides, but that he went so far.

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For ofcourse there is no reason in principle why one should not go on. Various people did, including:

Ptolemy	(c. 150 AD)	3.1416
Zu Chongzhi	(430-501 AD)	$^{355}/_{113}$
al-Khwarizmi	(c. 800)	3.1416
al-Kashi	(c. 1430)	14 places
Viète	(1540-1603)	9 places
Roomen	(1561-1615)	17 places
Van Ceulen	(c. 1600)	35 places

Except for Zu Chongzhi, about whom next to nothing is known and who is very unlikely to have known about Archimedes' work, there was no theoretical progress involved in these improvements, only greater stamina in calculation. Notice how the lead, in this as in all scientific matters, passed from Europe to the East for the millennium 400 to 1400 AD.

Al-Khwarizmi lived in Baghdad, and incidentally gave his name to 'algorithm', while the words al jabr in the title of one of his books gave us the word 'algebra'. Al-Kashi lived still further east, in Samarkand, while Zu Chongzhi, one need hardly add, lived in China.

The European Renaissance brought about in due course a whole new mathematical world. Among the first effects of this reawakening was the emergence of mathematical formulae for π. One of the earliest was that of Wallis (1616-1703)

$$2/\pi = (1.3.3.5.5.7. \ldots)/(2.2.4.4.6.6. \ldots).$$

and one of the best-known is

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$$

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This formula is sometimes attributed to Leibniz (1646-1716) but is seems to have been first discovered by James Gregory (1638- 1675).

These are both dramatic and astonishing formulae, for the expressions on the right are completely arithmetical in character, while p arises in the first instance from geometry. They show the surprising results that infinite processes can achieve and point the way to the wonderful richness of modern mathematics.

From the point of view of the calculation of p, however, neither is of any use at all. In Gregory's series, for example, to get 4 decimal places correct we require the error to be less than $0.00005 = \frac{1}{20000}$, and so we need about 10000 terms of the series. However, Gregory also showed the more general result

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots (-1 \le x \le 1) \quad \ldots (3)$$

from which the first series results if we put x = 1. So using the fact that

$$\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6} \text{ we get}$$

$$\frac{\pi}{6} = \left(\frac{1}{\sqrt{3}}\right)\left(1 - \frac{1}{(3.3)} + \frac{1}{(5.3.3)} - \frac{1}{(7.3.3.3)}\right) + \ldots$$

which converges much more quickly. The 10th term is $\frac{1}{\left(19 \times 3^9 \sqrt{3}\right)}$, which is less than 0.00005, and so we have at least 4 places correct after just 9 terms.

An even better idea is to take the formula

$$\frac{\pi}{4} = \tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{3}\right) \quad \ldots (4)$$

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and then calculate the two series obtained by putting first $\frac{1}{2}$ and the $\frac{1}{3}$ into (3).

Clearly we shall get very rapid convergence indeed if we can find a formula something like

$$\frac{\pi}{4} = \tan^{-1}\left(\frac{1}{a}\right) + \tan^{-1}\left(\frac{1}{b}\right)$$

with a and b large. In 1706 Machin found such a formula:

$$\frac{\pi}{4} = 4\tan^{-1}\left(\frac{1}{5}\right) - \tan^{-1}\left(\frac{1}{239}\right) \quad \ldots (5)$$

Actually this is not at all hard to prove, if you know how to prove (4) then there is no real extra difficulty about (5), except that the arithmetic is worse. Thinking it up in the first place is, of course, quite another matter.

With a formula like this available the only difficulty in computing p is the sheer boredom of continuing the calculation. Needless to say, a few people were silly enough to devote vast amounts of time and effort to this tedious and wholly useless pursuit. One of them, an Englishman named Shanks, used Machin's formula to calculate p to 707 places, publishing the results of many years of labour in 1873. Shanks has achieved immortality for a very curious reason which we shall explain in a moment.

Here is a summary of how the improvement went:

1699: Sharp used Gregory's result to get 71 correct digits

1701: Machin used an improvement to get 100 digits and the following used his methods:

1719: de Lagny found 112 correct digits

1789: Vega got 126 places and in 1794 got 136

1841: Rutherford calculated 152 digits and in 1853 got 440

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1873: Shanks calculated 707 places of which 527 were correct.

A more detailed Chronology is available.

Shanks knew that p was irrational since this had been proved in 1761 by Lambert. Shortly after Shanks' calculation it was shown by Lindemann that p is transcendental, that is, p is not the solution of any polynomial equation with integer coefficients. In fact this result of Lindemann showed that 'squaring the circle' is impossible. The transcendentality of π implies that there is no ruler and compass construction to construct a square equal in area to a given circle.

Very soon after Shanks' calculation a curious statistical freak was noticed by De Morgan, who found that in the last of 707 digits there was a suspicious shortage of 7's. He mentions this in his Budget of Paradoxes of 1872 and a curiosity it remained until 1945 when Ferguson discovered that Shanks had made an error in the 528th place, after which all his digits were wrong. In 1949 a computer was used to calculate p to 2000 places. In this and all subsequent computer expansions the number of 7's does not differ significantly from its expectation, and indeed the sequence of digits has so far passed all statistical tests for randomness.

You can see 2000 places of π.

We should say a little of how the notation p arose. Oughtred in 1647 used the symbol d/p for the ratio of the diameter of a circle to its circumference. David Gregory (1697) used p/r for the ratio of the circumference of a circle to its radius. The first to use p with its present meaning was an Welsh mathematician William Jones in 1706 when he states "3.14159 andc. = π". Euler adopted the symbol in 1737 and it quickly became a standard notation.

We conclude with one further statistical curiosity about the calculation of p, namely Buffon's needle experiment. If we have a uniform grid of parallel lines, unit distance apart

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and if we drop a needle of length k < 1 on the grid, the probability that the needle falls across a line is 2k/p. Various people have tried to calculate p by throwing needles. The most remarkable result was that of Lazzerini (1901), who made 34080 tosses and got

$$\pi = \frac{355}{113} = 3.1415929$$

which, incidentally, is the value found by Zu Chongzhi. This outcome is suspiciously good, and the game is given away by the strange number 34080 of tosses. Kendall and Moran comment that a good value can be obtained by stopping the experiment at an optimal moment. If you set in advance how many throws there are to be then this is a very inaccurate way of computing π. Kendall and Moran comment that you would do better to cut out a large circle of wood and use a tape measure to find its circumference and diameter.

Still on the theme of phoney experiments, Gridgeman, in a paper which pours scorn on Lazzerini and others, created some amusement by using a needle of carefully chosen length k = 0.7857, throwing it twice, and hitting a line once. His estimate for π was thus given by

$$\frac{2 \times 0.7857}{\pi} = \frac{1}{2}$$

from which he got the highly creditable value of π = 3.1428. He was not being serious!

It is almost unbelievable that a definition of π was used, at least as an excuse, for a racial attack on the eminent mathematician Edmund Landau in 1934. Landau had defined p in this textbook published in GOttingen in that year by the, now fairly usual, method of saying that π/2 is the value of x between 1 and 2 for which cos x vanishes. This unleashed an academic dispute which was to end in Landau's dismissal from his chair at GOttingen. Bieberbach, an eminent number

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theorist who disgraced himself by his racist views, explains the reasons for Landau's dismissal:-

> *Thus the valiant rejection by the GOttingen student body which a great mathematician, Edmund Landau, has experienced is due in the final analysis to the fact that the un-German style of this man in his research and teaching is unbearable to German feelings. A people who have perceived how members of another race are working to impose ideas foreign to its own must refuse teachers of an alien culture.*

G H Hardy replied immediately to Bieberbach in a published note about the consequences of this un-German definition of π :

> *There are many of us, many Englishmen and many Germans, who said things during the War which we scarcely meant and are sorry to remember now. Anxiety for one's own position, dread of falling behind the rising torrent of folly, determination at all cost not to be outdone, may be natural if not particularly heroic excuses. Professor Bieberbach's reputation excludes such explanations of his utterances, and I find myself driven to the more uncharitable conclusion that he really believes them true.*

Not only in Germany did p present problems. In the USA the value of p gave rise to heated political debate. In the State of Indiana in 1897 the House of Representatives unanimously passed a Bill introducing a new mathematical truth.

> *Be it enacted by the General Assembly of the State of Indiana: It has been found that a circular area is to the square on a line equal to the quadrant of the circumference, as the area of an equilateral rectangle is to the square of one side. (Section I, House Bill No. 246, 1897)*

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The Senate of Indiana showed a little more sense and postponed indefinitely the adoption of the Act!

Open Questions about the Number π

1. Does each of the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 each occur infinitely often in π?
2. Brouwer's question: In the decimal expansion of π, is there a place where a thousand consecutive digits are all zero?
3. Is p simply normal to base 10? That is does every digit appear equally often in its decimal expansion in an asymptotic sense?
4. Is p normal to base 10? That is does every block of digits of a given length appear equally often in its decimal expansion in an asymptotic sense?
5. Is p normal ? That is does every block of digits of a given length appear equally often in the expansion in every base in an asymptotic sense? The concept was introduced by Borel in 1909.
6. Another normal question! We know that π is not rational so there is no point from which the digits will repeat. However, if π is normal then the first million digits 314159265358979... will occur from some point. Even if π is not normal this might hold! Does it? If so from what point? Note: Up to 200 million the longest to appear is 31415926 and this appears twice.

As a postscript, here is a mnemonic for the decimal expansion of π. Each successive digit is the number of letters in the corresponding word.

How I want a drink, alcoholic of course, after the heavy lectures involving quantum mechanics. All of thy geometry, Herr Planck, is fairly hard...:

3.14159265358979323846264...

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You can see more about the history of p in the History topic: Squaring the circle and you can see a Chronology of how calculations of p have developed over the years.

Squaring the Circle

There are three classical problems in Greek mathematics which were extremely influential in the development of geometry. These problems were those of squaring the circle, doubling the cube and trisecting an angle. Although these are closely linked, we choose to examine them in separate articles. The present article studies what has become the most famous for these problems, namely the problem of squaring the circle or the quadrature of the circle as it is sometimes called.

One of the fascinations of this problem is that it has been of interest throughout the whole of the history of mathematics. From the oldest mathematical documents known up to the mathematics of today the problem and related problems concerning p have interested both professional mathematicians and amateur mathematicians.

You can see more about the history of π in the History topics: Pi through the ages and A chronology of Pi.

One of the oldest surviving mathematical writings is the Rhind papyrus, named after the Scottish Egyptologist A Henry Rhind who purchased it in Luxor in 1858. It is a scroll about 6 metres long and 1/3 of a metre wide and was written around 1650 BC by the scribe Ahmes who copied a document which is 200 years older. This gives date for the original papyrus of about 1850 BC but some experts believe that the Rhind papyrus is based on a work going back to 3400 BC.

You can see more about the Rhind papyrus in the History topic article Egyptian papyri.

In the Rhind papyrus Ahmes gives a rule to construct a square of area nearly equal to that of a circle. The rule

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is to cut $\frac{1}{9}$ off the circle's diameter and to construct a square on the remainder. Although this is not really a geometrical construction as such it does show that the problem of constructing a square of area equal to that of a circle goes back to the beginnings of mathematics. This is quite a good approximation, corresponding to a value of 3.1605, rather than 3.14159, for π.

The problem of squaring the circle in the form which we think of it today originated in Greek mathematics and it is not always properly understood. The problem was, given a circle, to construct geometrically a square equal in area to the given circle. The methods one was allowed to use to do this construction were not entirely clear, for really the range of methods used in geometry by the Greeks was enlarged through attempts to solve this and other classical problems. Pappus, writing in his work Mathematical collection at the end of the period of Greek development of geometry, distinguishes three types of methods used by the ancient Greeks:-

> *There are, we say, three types of problem in geometry, the so-called 'plane', 'solid', and 'linear' problems. Those that can be solved with straight line and circle are properly called 'plane' problems, for the lines by which such problems are solved have their origin in a plane. Those problems that are solved by the use of one or more sections of the cone are called 'solid' problems. For it is necessary in the construction to use surfaces of solid figures, that is to say, cones. There remain the third type, the so-called 'linear' problem. For the construction in these cases curves other than those already mentioned are required, curves having a more varied and forced origin and arising from more irregular surfaces and from complex motions.*

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Now we usually think of the problem of squaring the circle to be a problem which has to be solved using a ruler and compass. This is really asking whether squaring the circle is a 'plane' problem in the terminology of Pappus given above (we shall often refer to a 'plane solution' rather than use the more cumbersome 'solutions using ruler and compass"). The ancient Greeks, however, did not restrict themselves to attempting to find a plane solution (which we now know to be impossible), but rather developed a great variety of methods using various curves invented specially for the purpose, or devised constructions based on some mechanical method.

The first mathematician who is on record as having attempted to square the circle is Anaxagoras. Plutarch, in his work On Exile which was written in the first century AD, says :-

> *There is no place that can take away the happiness of a man, nor yet his virtue or wisdom. Anaxagoras, indeed, wrote on the squaring of the circle while in prison.*

Now the problem must have become quite popular shortly after this, not just among a small number of mathematicians, but quite widely, since there is a reference to it in a play Birds written by Aristopenes in about 414 BC. Two characters are speaking, Meton is the astronomer (see D Barrett (trs.), Aristophanes, Birds (London, 1978) for a shorter quote):-

Meton: I propose to survey the air for you: it will have to be marked out in acres.

Peisthetaerus: Good lord, who do you think you are?

Meton: Who am I? Why Meton. THE Meton. Famous throughout the Hellenic world - you must have heard of my hydraulic clock at Colonus?

Peisthetaerus (eyeing Meton's instruments): And what are these for?

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Meton: Ah! These are my special rods for measuring the air. You see, the air is shaped - how shall I put it? - like a sort of extinguisher: so all I have to do is to attach this flexible rod at the upper extremity, take the compasses, insert the point here, and - you see what I mean?

Peisthetaerus: No.

Meton: Well I now apply the straight rod - so - thus squaring the circle: and there you are. In the centre you have your market place: straight streets leading into it, from here, from here, from here. Very much the same principle, really, as the rays of a star: the star itself is circular, but sends out straight rays in every direction.

Peisthetaerus: Brilliant - the man's a Thales.

Now from this time the expression 'circle-squarers' came into usage and it was applied to someone who attempts the impossible. Indeed the Greeks invented a special word which meant 'to busy oneself with the quadrature'. For references to squaring the circle to enter a popular play and to enter the Greek vocabulary in this way, there must have been much activity between the work of Anaxagoras and the writing of the play. Indeed we know of the work of a number of mathematicians on this problem during this period: Oenopides, Antiphon, Bryson, Hippocrates, and Hippias.

Oenopides is thought by Heath to be the person who required a plane solution to geometry problems. Proclus attributes two theorems to Oenopides , namely to draw a perpendicular to a line from a given point not on the line, and to construct from a given point on a given line, a line at a given angle to the given line. Heath believes that the significance of these elementary results was that Oenopides set out for the first time the explicit 'plane' or 'ruler and compass' type of construction. Heath writes :-

... [Oenopides] may have been the first to lay down the restriction of the means permissible in

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constructions t the ruler and compasses which became a canon of Greek geometry for all plane constructions...

There is no record of any attempt by Oenopides to square the circle by plane methods. In fact it is a rather remarkable fact that the Greeks did not produce fallacious 'proofs' that the circle could be squared by plane methods. The few claims for such false proofs rather seem to result from less able mathematicians failing to understand exactly what some of the more brilliant contributions to the problem were intended to show.

Sadly later mathematicians did not follow the good example shown by the ancient Greeks and indeed many claimed incorrectly to have discovered a 'ruler and compass' proof. Amateur mathematicians, greatly attracted to the classical problems, have produced (and still continue to produce) thousands of false proofs.

Antiphon and Bryson both produced arguments relating to squaring the circle which were to prove important in the future development of mathematics. Antiphon inscribed a square in a circle, then a regular polygon with eight sides, then one with sixteen sides and he continued the process continually doubling the number of sides. It appears that Bryson improved the argument of Antiphon by not only inscribing polygons in a circle but also circumscribed polygons. Themistius states :-

... that Bryson declared the circle to be greater than all inscribed, and less than all circumscribed polygons.

Hippocrates was the first to actually use a plane construction to find a square with area equal to a figure with circular sides. He squared certain lunes, and also the sum of a lune and a circle.

Now although he squared certain lunes, he had not shown that every lune can be squared. In particular the lune that he squared in his plane construction of a square of area

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equal to that of a certain lune and a circle was one he could not square by plane methods. Of course this lune cannot be squared by plane methods otherwise Hippocrates would have squared the circle. Although some, such as Aristotle, seemed to fail to understand the logic of Hippocrates argument, there seems little doubt that Hippocrates was perfectly aware that his methods had failed to square the circle. Examples of Hippocrates' methods of squaring lunes are given in his biography in this archive.

Hippias and Dinostratus are associated with the method of squaring the circle using a quadratrix. The curve it thought to be the invention of Hippias while its application to squaring the circle appears to be due to Dinostratus. The construction of this curve with a diagram is given in the biography of Hippias in this archive. Now this curve certainly solves the problem of squaring the circle but, as given by Hippias, the curve is constructed by mechanical means given by a uniform motion of a line in a time equal to the rotating radius of a circle. The construction was rightly criticised as requiring a knowledge of the ratio of a line and an arc of a circle, so one assumed as known the property required to square the circle in the first place. It is clear that Dinostratus never claimed that the quadratrix gave a plane method to square the circle. Nicomedes many years later also used the quadratrix to square the circle.

Aristotle did not seem to appreciate the contributions of those who had attempted to square the circle. He wrote in his work Physics :-

> *The exponent of any science is not called upon to solve every kind of difficulty that may be raised, but only such as arise through false deductions from the principles of the science: with others than these he need not concern himself. For example, it is for the geometer to expose the quadrature by means of segments, but it is not the business of the geometer to refute the arguments of Antiphon.*

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In this quote "quadrature by means of segments" refers to Hippocrates quadrature of lunes which Aristotle mistakenly thinks was intended as a proof that the circle can be squared by plane methods. Antiphon's methods come in for even more criticism from Aristotle, but all credit to Antiphon whose methods contained important ideas which would lead eventually to integration. Aristotle also wrote in similar terms in Sophistical refutations again probably having had handed down to him an incorrect interpretation of what Antiphon and Bryson had attempted to show:-

> *The method by which Bryson tried to square the circle, were it ever so much squared thereby, is yet made sophistical by the fact that it has no relation to the matter in hand. ... The squaring of the circle by means of lunes is not eristic, but the quadrature of Bryson is eristic. The reasoning used by the former cannot be applied to any subject other than geometry alone, whereas Bryson's argument is directed to the mass of people who do not know what is possible and what is impossible in each department, for it will fit any. And the same is true of Antiphon's quadrature.*

Next we should consider the contributions of Archimedes to the problem of squaring the circle. Now Archimedes is famed for his introduction of the spiral curve, but why did he introduced this curve? The authors suggest three reasons:-

> *Is it for purely geometric reasons because he studied this curve as a means of calculating p, and squaring the circle? Is it because of his astronomical interests, trying to calculate geometrically the spiral movements of the planets? Or is it finally through the interest of a mechanical mind in a curve which results from the combination of two regular uniform movements, one in a straight line the other in a circle? these three reasons are evident at one and the same time...*

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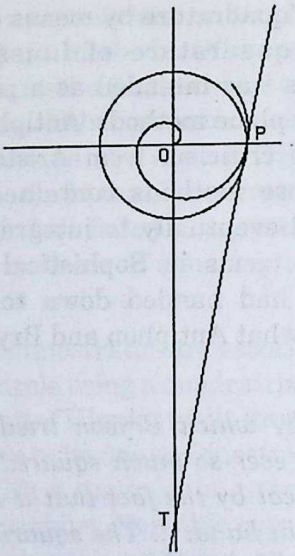

Archimedes gives the following definition of the spiral in his work On spirals:-

> *If a straight line drawn in a plane revolves uniformly any number of times about a fixed extremity until it returns to its original position, and if, at the same time as the line revolves, a point moves uniformly along the straight line beginning at the fixed extremity, the point will describe a spiral in the plane.*

To square the circle Archimedes gives the following construction. Let P be the point on the spiral when it has completed one turn. Let the tangent at P cut the line perpendicular to OP at T. Then Archimedes proves in Proposition 19 of On spirals that OT is the length of the circumference of the circle with radius OP.

Now it may not be clear that this is solved the problem of squaring the circle but Archimedes had already proved as the first proposition of Measurement of the circle that the area of a circle is equal to a right-angled triangle having the two shorter sides equal to the radius of the circle and the

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circumference of the circle. So the area of the circle with radius OP is equal to the area of the triangle OPT.

Both Apollonius and Carpus used curves to square the circle but it is not clear exactly what these curves were. The one used by Apollonius is called by Iamblichus 'sister of the cochloid' and this has led to various guesses as to what the curve might have been. Again the curve used by Carpus of Antioch is called the 'curve of double motion' which Paul Tannery argued was the cycloid.

Now we leave the ancient Greek period and look at later developments but the first comment we should make is that the Greeks were certainly not the only ones to be interested in squaring the circle at this time. Mathematicians in India were interested in the problem while in China mathematicians such as Liu Hsiao of the Han Dynasty showed himself to be one of the prominent of those attempting to square the circle in around 25 AD.

Some time later the Arab mathematicians were, like the Greeks, fascinated by the problem. The work of al-Haytham on squaring the circle is discussed. Now al-Haytham aimed to convince people that squaring the circle was possible by a plane construction but since his promised treatise on the topic never appeared he must at least have realised that he could not solve the problem.

Not long after the work of al-Haytham, Franco of Liège in 1050 wrote a treatise De quadratura circuli on squaring the circle. The text is reproduced and in it Franco examines three earlier methods based on the assumption that p is 25/8 , 49/16 or 4. Franco states (reasonably enough) that these are false, then gives his own construction which is based on the assumption that p is 22/7 .

Although this treatise is of great historical interest, it does show how European mathematics at the time was far behind the ancient Greeks in depth of understanding.

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Moving forward to about 1450, Cusa attempted to prove that the circle could be squared by a plane construction. Although his method of averaging certain inscribed and circumscribed polygons is quite fallacious, it is one of the first serious attempts in 'modern' Europe to solve the problem. Again it is worth commenting that the ancient Greeks basically knew that the circle could not be squared by plane methods, although they stood no chance of proving it. Regiomontanus, who brought a new impetus to European mathematics, was quick to point out the error in Cusa's arguments.

The mechanical methods of the Greeks certainly appealed to Leonardo who thought about mathematics in a very mechanical way. He devised several new mechanical methods to square the circle. Many mathematicians in the sixteenth century studied the problem, including Oronce Fine and Giambattista della Porta. The 'proof' by Fine was shown to be incorrect by Pedro Nunes soon after he produced it. The beginnings of the differential and integral calculus led to an increased interest in squaring the circle, but the new era of mathematics still produced fallacious 'proofs' of plane methods to square the circle. One such false proof, given by Saint-Vincent in a book published in 1647, was based on an early type of integration. The problem was still providing much impetus for mathematical development.

James Gregory developed a deep understanding of infinite sequences and convergence. He applied these ideas to the sequences of areas of the inscribed and circumscribed polygons of a circle and tried to use the method to prove that there was no plane construction for squaring the circle. His proof essentially attempted to prove that p was transcendental, that is not the root of a rational polynomial equation. Although he was correct in what he tried to prove, his proof was certainly not correct. However, others such as Huygens, believed that p was algebraic, that is that it is the root of a rational polynomial equation.

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There was still an interest in obtaining methods to square the circle which were not plane methods. For example Johann Bernoulli gave a method of squaring the circle through the formation of evolvents and this method is described .

The historian of mathematics, Montucla, made squaring the circle the topic of his first historical work published in 1754. This was written at a time long before the problem was finally resolved, so is necessarily very outdated. The work is, however, a classic and still well worth reading.

A major step forward in proving that the circle could not be squared using ruler and compasses occurred in 1761 when Lambert proved that p was irrational. This was not enough to prove the impossibility of squaring the circle with ruler and compass since certain algebraic numbers can be constructed with ruler and compass. It only led to a greater flood of amateur solutions to the problem of squaring the circle and in 1775 the Paris Académie des Sciences passed a resolution which meant that no further attempted solutions submitted to them would be examined. A few years later the Royal Society in London also banned consideration of any further 'proofs' of squaring the circle as large numbers of amateur mathematicians tried to achieve fame by presenting the Society with a solution. This decision of the Royal Society was described by De Morgan about 100 years later as the official blow to circle-squarers.

The popularity of the problem continued and there are many amusing stories told by De Morgan on this topic in his book Budget of Paradoxes which was edited and published by his wife in 1872, the year after his death. De Morgan suggests that St. Vitus be made the patron saint of circle-squarers. This is a reference to St. Vitus' dance, a wild leaping dance in which people screamed and shouted and which led to a kind of mass hysteria. De Morgan also suggested the term 'morbus cyclometricus' as being the 'circle squaring disease'. Clearly De Morgan found himself having

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to try to persuade these circle-squarers that their methods were incorrect, yet many stubbornly held to their views despite the best efforts of the professional mathematicians. For example a certain Mr James Smith wrote several books attempting to prove that $\pi = \frac{25}{8}$. . Of course Mr Smith was able to deduce from this that the circle could be squared but neither Hamilton, De Morgan nor others could convince him of his errors.

The final solution to the problem of whether the circle could be squared using ruler and compass methods came in 1880 when Lindemann proved that p was transcendental, that is it is not the root of any polynomial equation with rational coefficients. The transcendentality of p finally proves that there is no ruler and compass construction to square the circle.

One might imagine that this would be the end of interest in the problem of squaring the circle, but this was certainly not the case. It neither prevented the stream of publications claiming that p had some simple rational value, nor did it prevent the stream of publications of quite correct constructions to approximately square the circle with ruler and compass. As an example of the former type of claim, the New York Tribune published a letter in 1892 in which the author claimed to have rediscovered a secret going back to Nicomedes which proved that $\pi = 3.2$. Perhaps more surprising is the fact that there were many who were totally convinced by this letter and firmly believed thereafter that $\pi = 3.2$.

Among the correct approximate constructions to square the circle was one by Hobson in 1913. This was a fairly accurate construction which was based on constructing the approximate value of 3.14164079... for p instead of 3.14159265.... . More remarkable, however, was the ruler and compass constructions published by Ramanujan. In the

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Journal of the Indian Mathematical Society in 1913 in a paper named Squaring the circle Ramanujan gave a construction which was equivalent to giving the approximate value of 355/113 for p, which differs from correct value only in the seventh decimal place. He ended the paper with the following:-

Note. If the area of the circle be 140,000 square miles, then [the side of the square] is greater than the true length by about an inch.

Among other constructions given by Ramanujan in 1914 (Approximate geometrical constructions for p, Quarterly Journal of Mathematics XLV (1914), 350-374) was a ruler and compass construction which was equivalent to taking the strange yet remarkable approximate value for π to be $(9^2 + 19^2/22)^{1/4}$. Now this is 3.1415926525826461253.... which differs from π only in the ninth decimal place (π = 3.14159265358979323 85...). For a circle of diameter 8000 miles, the error in the length of the side of the square constructed was only a fraction of an inch.

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2

FAQ in Ancient Indian Mathematics

Strengths of the Ancient Indian Mathematicians

The Indian Mathematicians of the ancient era were primarily number theorists. Their interest in this field arose from a need to do astronomical calculations, which in turn were needed to advise the farmers on the proper timing to plant crops and for performing rituals . In particular they excelled in Diophantine Equations (probably erroneously credited to Diophantus , because the Vedics were adapt at it long before Diophantus), algebraic equations in which only solutions in integers are permitted. Examples of Diophantine equations that the ancients dealt with are

- $ax + by = 1$: this is a linear Diophantine.
- $x^n + y^n = z^n$: For $n = 2$ there are infinitely many solutions (x,y,z), the Pythagorean triples. For larger values of n, Fermat's last theorem states that no positive integer solutions x, y, z satisfying the above equation exist.
- $x^2 - n\ y^2 = 1$: (Pell's equation) which is named, mistakenly, after the English mathematician John Pell. It was studied by Brahmagupta and much later by Fermat.

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Another field or approach which the ancients favoured was algebraic geometry, where geometry was studied predominantly using algebraic equations. Again the need was for designing ritual altars and their proper orientation with respect to the heavenly bodies and for following the rules of Vastu shastra (architecture) .

The use of concise mathematical symbolism (<, >, = ,*, (), inf., sup., the integral sign, the derivative sign, the sigma sign etc. is a relatively recent revolution in mathematics, When calculus was invented by Newton and Leibniz independently, they used different notations. Newton used dots on top of the quantity and called them fluxions, while Leibniz used the 'd' notation. Unfortunately when all the major advancements in science and mathematics were happening in Europe in the 17th and 18th century, India was stagnant, caught in interminable wars and conquests.

Ironically it was the Indian place value notation that triggered the advance in Europe. Prior to this it was common to express all problems in wordy sentences and verse.

As to the value of PI (cannot be expressed as a fraction) it is impossible to calculate it without the use of some variant of the 'method of exhaustion'. In this particular instance it was a matter of using increasingly larger number of isosceles triangles(forming an n sided polygon) within the circle. This is the germ of the method of series expansions.

The ancient Indians were well aware of this technique and used it for several trigonometric calculations. The felicity with which Indians did series expansions extended to Srinivasa Ramanujam in recent times. He was the incomparable master and there may be none like him on the face of this earth again.

Again there is no claim that the Indians did everything. For example there is no evidence that the Indians were

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familiar with the representation of complex numbers and complex variables or advanced topics such as the calculus of variations. The ancient vedic Indics were interested in very practical aspects of mathematics, namely the positions of the stars, developing a Panchanga (calendar), ordinary mathematics for everyday use, measurements, such as that of land and weights etc.. There is no evidence that they were familiar with the science of mechanics for instance, which developed in Europe in the 18th century after Newton.

Of course, the most important single concept that they developed was the decimal place value system. While other civilizations like the Maya and the Oaxaca of ancient Mexico developed place value systems independently, there is no evidence that they mastered the technique of using numbers with such facility as the Vedics did. The Vedics also developed facility with areas such as Trigonometry and there is mention that they manipulated arrays of numbers in the same manner as we use matrices today.

Even with all these skills, it must be recognized that this forms only a small part of what constitutes the corpus of modern mathematics today. It must also be recognized that the Greeks and the Babylonians also had developed mathematical prowess, if one accepts that the axiomatic approach to geometry as primarily a Greek development.

Despite all these caveats, it must be admitted that for the era in which they lived the Vedic contributions to the sum of human knowledge especially in mathematics was considerable and should therefore be a matter of great inspiration for those of us who consider themselves part of the Indic civilization.

Further there is much study yet to be done and many more manuscripts which remain uninvestigated so the quest for what was the state of knowledge of the ancients in fields such as mathematics has just begun.

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Use the Term Vedic Mathematics

Strictly speaking the later contributors like Varahamihira did not live in the Vedic era, but the methods used show a continuity in development till the Modern age and hence the use of the term Vedic , signifying techniques which came into use in the distant past seems justified.

Vedic Mathematics Progressed much Farther in the Intervening Millennia?

I will quote the answer I gave in a discussion forum (Bharat-Rakshak)a few years ago.

First, do we understand the corpus of VM and Vedic Science (i certainly don't know enough about VM to make an authoritative statement). What part of what we know to be contributions, were purely Vedic and what part were contributions by mathematicians like Aryabhatta and Bhaskaracharya who came much later ?

Second what part of it is already accepted in modern mathematics (and I am not talking about arithmetical tricks to do multiplication and division). I remember in my Hall and Stevens text on geometry which i used in India, the proof of Pythagoras theorem took up a whole page. I would have loved to have the 4 alternate proofs offered by the VMB. I am interested for example in the development of Astronomy.

Ptolemy used a concept called ecliptics (if i recall the use of epicycles - pl. don't harangue me on this, as this is of the top of my head) to get around the fact that the earth went around the sun and not the other way around. Did he borrow that from Vedic or was that a different stream of thought (Ptolemy of course predates al-Khwarismi and his 'Sindhind zij'). Given that Ptolemy post dates Vedic why did al-Khwarismi choose to rely more on Vedic (a more ancient technology) when he presumably had access to both ?

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Third the point made by James is valid. The point is not to judge Vedic Science by the standards of 21st century and then trash it saying it is outmoded. Of course it is outmoded, in certain respects, you should expect that after 6 millennia. If it was not, that is tantamount to admitting that we have made no progress in the intervening millennia. The point is, does it give an alternate model at looking at nature (e.g. Ayurveda) that is equally valid, useful and perhaps more elegant.

Fourth, these efforts at understanding our past, should not be restricted to Vedic systems but to other systems in India's past such as Yunani (etymology Ionian = Greek) which is attributed to Islamic savants. I am interested for example in tracing through the development of algebra (coined by al Khwarismi as al-gibr wal Maquaballah) from its ancient Vedic origins because of the efforts of Islamic savants in the middle ages, at a time when Europe was in the dark ages and was struggling with Roman numerals.

My own view is that Vedic science and math is a forgotten science and for the most part does not contradict what has been discovered subsequently. First let us understand what it says before getting an anxiety neurosis that it is going to replace Western science in Indian schools.

Results to be Found in the Ancient Texts

The earliest works on Mathematics by the Vedic savants are recorded in the Sulvasutras, the sacred books of altar construction in the Vedas, in particular the Apastambha Sulvasutra, the Baudhayana Sulvasutra and the Katyayana Sulvasutra (named after the mathematicians who developed the mathematics and wrote the sutras. The Sulvasutras are appended to a particular Veda (see FAQ on Hinduism for the typical contents of a Veda) can be translated as rope rules or "manuals of measurement", the modern term for which would be metrology. But in reality the scope of the

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investigations in the Sulvasutras is far broader and comprises among other fields, number theory, trigonometric, algebra, algebraic geometry, series expansions, the concept of a rational number, etc., etc.. The dating of these Sulvasutras, while occurring after the main corpus of the Veda was compiled, is of great antiquity, greater than that of Babylonian mathematics. It is only now that we are beginning to understand the extent of the antiquity of these ancient mathematicians. See for instance The origin of Mathematics by Lakshmikantham and Leela, in the list of references later in this page.

A word needs to be said about the use of Sutras (Aphorisms) as a means of communication and recording of results. The dictionary says that a sutra is Any of various aphoristic doctrinal summaries produced for memorization generally during the millennia before the common era and later incorporated into Hindu literature. We must recall that writing materials during that period of history, were not plentiful and had to be laboriously produced probably by the author of the sutras himself . Whatever needed to be communicated had to be in as brief a form as possible. Hence the need for economy in language and the use of sutras. In fact there is evidence from which one can infer that the Vedics were the first to use symbols and mathematical equations and hence their prowess with Algebra .

Brevity has its downside however and the charge has been made by Europeans that the Vedics rarely provided proof. The reason was that the proof was generally terse and incomprehensible to the majority of the people. The notion that those who mastered these topics used brevity as a means to restricting this knowledge, overlooks the fact few in the populace would have the capability to pursue this rigorously without the intellectual discipline that comes from years of study. Then as now Mathematics had the reputation of being a difficult subject to master.

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Main Contributors to Vedic Mathematics

The main contributors during the Vedic period were, as we mentioned earlier Baudhayana, Apastambha, and Katyayana. The following list of Mathematicians who were born in the geographical region corresponding to present day India was compiled by researchers at the University of St. Andrews, Scotland, who have done an outstanding job in compiling a lot of information (this list includes some modern day mathematicians born in the Indian sub continent also).

Contributions of the Vedic Savants not as Widely known as those of Greek, Arab Mathematicians

In the early years of colonial rule by the British (the attitude persists among Western Indologists even today, although less so among mathematicians in the West) there was great reluctance to believe the sacred texts, after they were first recognized by Sir William Jones in the 1770's. A typical reaction was that of W. W Rouse Ball in History of Mathematics . I posted the following in the Bharat Rakshak forum in 2000.

"Typical of the racism exhibited by the Brits and other Europeans is W.W. Rouse Ball in 'A short account of the History of mathematics' Dover Publications, 1960, (originally appeared in 1908), page 146 'The Arabs had considerable commerce with India, and a knowledge of one or both of the two great Hindu works on algebra had been obtained in the Caliphate of Al-Mansur (754-775 AD) though it was not until fifty or seventy years later that they attracted much attention. The algebra and arithmetic of the Arabs were largely founded on these treatises, and I therefore devote this section to the consideration of Hindu mathematics. The Hindus like the Chinese have pretended that they are the most ancient people on the face of the earth, and that to them all sciences owe their creation. But it is probable that these pretensions have no foundation; and in fact no science

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or useful art (except a rather fantastic architecture and sculpture) can be definitely traced back to the inhabitants of the Indian peninsula prior to the Aryan invasion. This seems to have taken place at some time in the fifth century or in the sixth century when a tribe of Aryans entered India by the north west part of their country. Their descendants, wherever they have kept their blood pure, may still be recognized by their superiority over the races they originally conquered; but as is the case with the modern Europeans, they found the climate trying and gradually degenerated. Note the blatant racism in the second paragraph and the venom that this author exhibits. [This message has been edited by Kaushal (edited 15-06-2000).]

There are a lot of facile and unsubstantiated assumptions that are made here, later to be proved false, but we will deal with them on the chapter where we discuss AIT. Thus there was great reluctance to admit that the dark skinned natives of the Indian subcontinent could be capable of intellectual effort. Even after the advent of the legendary Srinivasa Ramanujam, the great number theorist in the early years of the 20th century, from what is now Chennai, Tamil Nadu , such attitudes among British and European scholars were hard to dispel. We will have a lot more to say about Srinivasa Ramanujam later in these pages.

With the coming of the internet, and the great proficiency of the Indics in matters related to Information Technology, this state of affairs has begun to change. Both the Indics and Western savants have begun to realize the profound importance of these early developments in mathematics to the advancement of human civilization. See for instance a recent column on Place Value systems.

Can we see Examples of what you Mean by Vedic Mathematics

First we will make a distinction between the corpus of Vedic Mathematics (VM) from the 16 extant sutras that are

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still available to us courtesy of the book on Vedic mathematics (VMB) by Jagadguru Bhaarathi Krishnatirtha, the late Shankaracharaya of Puri, here we will restrict ourselves to VMB, which is largely a text at High school level, although I will wager very few school children even in India are familiar with these techniques. So what does VMB comprise of?

The Natural Calculator

We have probably been taught methods, which require pencil and paper or calculator, to work out because of the difficulty and a lack of encouragement for mental calculation, which have prevented us form becoming efficient mental calculators. Though mathematics has applications at many levels, it is primarily a mental subject. A system of mental mathematics could provide students with a lasting link with the realm of mathematics and also engender a deeper understanding of the structure and processes of mathematics. The beautiful system of Vedic Mathematics is far more unified and direct than conventional mathematics. It is so easy that it is really a system of mental mathematics, and this combined with the coherence and flexibility of the system, encourages the development and use of intuition and creativity. These features make Vedic Mathematics very attractive. Vedic Mathematics deals with all the fields in Mathematics. The simplicity of Vedic Mathematics means that calculations can be carried out mentally and this is very much encouraged in the Vedic system. Being naturally creative students is to devise their own methods of solution. The Vedic system seeks to cultivate intuition having a conscious proof or explanation of a method beforehand is not considered essential in the Vedic methodology. Students are shown general methods and also special methods, which apply in special cases. This means they do not rigidly have to follow a certain procedure but have a choice and are encouraged to invent their own methods. Every problem is

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unique and invites its own style of solution. The Vedic system does not impose any unnecessary constraints. For e.g. we can work from left to right or vice-versa with both positive and negative numbers; more than two numbers can be operated simultaneously.

The Vedic system uses a collection of sixteen Sutras (or formulae), given in word form, and some sub-Sutras. They express fundamental principles, which run like threads ('Sutra' means 'thread') through the hole of mathematics, unifying diverse topics. They are extremely useful in education for this reason. We use our mind in certain specific ways - we might extend an idea or reverse it or compare or combine it with another.

Each of these types of mental activity is described by one of the Vedic Sutras. They describe the ways in which the mind can work and so they tell the student, how to go about solving a problem. It is not necessary for the teacher to be familiar with these. It is best not to stress the Sutras. They become familiar as you go through and seem quite natural. The best part of Vedic Mathematics is revealed in it's applications. Let us consider one of the applications using "Nikhilam Navatascaramam Dasatah". 'Nikhilam Navatascaramam Dasatah', literally translated in English, means 'All from nine and last from ten'. This means that we are supposed to take into consideration, the base to be that power of ten, which is closest to the number.

Everyone knows the universal fact that for multiplication and division, by the current method, a pupil of mathematics is forced to learn the multiplication tables of 20*20 and so on. But, according to Vedic System, the multiplication required to be learnt is maximum 5*5 and all that he has to know more than this is simple addition and subtraction (of single-digit numbers). So these formulae enable him to get at the required products, very easily and speedily, practically and immediately.

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The important applications of this Sutra are: Multiplication and Division. Since examples would make things clearer, let us consider multiplication first.

Say we have to multiply 9 by 7.

Here, we consider 10 (that power of ten, which is closest to the number to be multiplied - 9 - in this case). The modus-operandi chart is as shown, where the method is clearly understood.

The base will be that power of 10 which is closest to 9. Therefore the base is 10.	10
Put the numbers 9 and 7 above and below on the left-hand side.	9 7
Connecting with a minus sign on the right-hand side is the 10's complement of the corresponding left-hand side number.	9 - (10 - 9) = 9 - 1 7 - (10 - 7) = 7 - 3
The left digit is obtained by adding the above 2 left-hand side numbers and subtracting 10 from this sum.	A = (9 +7)-10 = 6

Vertical multiplication of the two deficit right-side numbers gives the right-side result.

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3

History of Vedic Mathematics

Calculating Prodigies

People who can perform astonishingly long or complex calculations in their head are often called calculating prodigies or lightning calculators. There are many examples from the past, from Jedediah Buxton who worked as a labourer and remained illiterate all his life to Karl Friedrich Gauss a mathematician who is ranked with Newton and Archimedes. Since Vedic Mathematics is the most efficient mathematical system it would be interesting to ask whether these calculating prodigies may have used it.

It is peculiar to arithmetic that once the basic facts are understood any intelligent person can construct the whole science: number science develops in a totally natural way. This is what seems to have happened in most cases of calculating prodigies. From a basic understanding the young calculator realises that the power of their mind to calculate seems to be almost unlimited. Once they understand a problem they can solve it. Ampere was an amazing mental calculator at the age of three and Gauss at the same age corrected his father's mistake in a payroll reckoning. These people are self-taught, they construct their own methods of calculation.

In some cases however the methods they devise are extraordinarily complicated, especially in view of the ultra-

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efficient methods of Vedic Mathematics. Jedediah Buxton, for example, could multiply huge numbers together in his head but seems to use the most awkward methods, multiplying parts of the number and adding them. This may not have been difficult for him as he had an extraordinary memory. He could remember many long numbers and recite them from the left, from the right or from any other point. He would also remember all the steps in a calculation for a long time afterwards.

All mental calculators have a good memory and they make use of previous results they have found out. Some also work two or more figures at a time (a feature of the Vedic system, as is calculating from right to left or left to right). Calculating prodigies do not seem to necessarily always seek the most efficient method - only one that works and that they are happy with. But calculations can often become unconscious and the calculator cannot then describe their method. And furthermore what may have been an effective method can get altered and simplified with practice so that many calculating prodigies may be using Vedic methods without knowing it.

When Bharati Krsna Tirthaji gave talks in India after discovering the Vedic system some people took advantage by learning the Vedic methods and giving demonstrations, sometimes pretending to be divinely inspired or to have the help of some spirit. This demonstrates the effectiveness of Vedic Mathematics as an efficient mental system. The Introduction to book "The Natural Calculator" describes the many advantages of mental mathematics and the Vedic system and supports the view that everyone is potentially a lightning calculator.

What is Vedic Mathematics?

Vedic Mathematics is the name given to the ancient system of Mathematics which was rediscovered from the

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Vedas between 1911 and 1918 by Sri Bharati Krishna Tirthaji (1884-1960). According to his research all of mathematics is based on sixteen Sutras or word-formulae. For example, 'Vertically and Crosswise' is one of these Sutras. These formulae describe the way the mind naturally works and are therefore a great help in directing the student to the appropriate method of solution.

Perhaps the most striking feature of the Vedic system is its coherence. Instead of a hotch-potch of unrelated techniques the whole system is beautifully interrelated and unified: the general multiplication method, for example, is easily reversed to allow one-line divisions and the simple squaring method can be reversed to give one-line square roots. And these are all easily understood. This unifying quality is very satisfying, it makes mathematics easy and enjoyable and encourages innovation.

In the Vedic system 'difficult' problems or huge sums can often be solved immediately by the Vedic method. These striking and beautiful methods are just a part of a complete system of mathematics which is far more systematic than the modern 'system'. Vedic Mathematics manifests the coherent and unified structure of mathematics and the methods are complementary, direct and easy.

The simplicity of Vedic Mathematics means that calculations can be carried out mentally (though the methods can also be written down). There are many advantages in using a flexible, mental system. Pupils can invent their own methods, they are not limited to the one 'correct' method. This leads to more creative, interested and intelligent pupils.

Interest in the Vedic system is growing in education where mathematics teachers are looking for something better and finding the Vedic system is the answer. Research is being carried out in many areas including the effects of learning Vedic Maths on children; developing new, powerful but easy applications of the Vedic Sutras in geometry,

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calculus, computing etc. But the real beauty and effectiveness of Vedic Mathematics cannot be fully appreciated without actually practising the system. One can then see that it is perhaps the most refined and efficient mathematical system possible.

Origin

The origin of "Magical Methods of Fast Calculations" is from Vedic sutras. There are sixteen Sutras and thirteen Sub-Sutras scattered in the writings of "Atharva Vedas" the fourth Veda. "Atharva Veda" deals with the branches like Engineering, Mathematics, sculpture, Medicine, and all other sciences with which we are today aware of.

The first book on these wonderful methods was written by Late Swami Bharati Krishna Tirtha ji Maharaj, Shankaracharya of Goverdhan Peeth. He was the person who collected lost formulae from the writings of "Atharwa Vedas" and wrote them in the form of Sixteen Sutras and thirteen sub-sutras.

These Sutras and Sub-Sutras can be applied to Arithmetical computations, theory of numbers, compound multiplication, algebraic operations, factorisations, simple quadratic and higher order equations, simultaneous quadratic equations, partial fractions, calculus, squaring, cubing, square root, cube root, coordinate geometry etc.

Magical Methods of Fast Calculations has been written in very simple and lucid language, which everybody can understand.

"India was the motherland of our race and Sanskrit the mother of Europe's languages. India was the mother of our philosophy, of much of our mathematics, of the ideals embodied in Christianity...of self-government and democracy. In many ways, Mother India is the mother of us all." — Will Durant, American Historian 1885-1981

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What is an ancient system of mathematics that is being taught in some of the most prestigious institutions in England and Europe and not in India? It is Vedic mathematics—a long forgotten technique for mathematical calculations! It is amazing how with the help of 16 sutras and 16 upa-sutras you will be able to solve/calculate complex mathematical problems mentally! The basic roots of Vedic mathematics lie in Vedas jus as basic roots of Hinduism. Vedic Maths form part of Jyothisha which is one of the six Vedangas. To many Indians Vedic and Sanskrit slokas/manthras are relevant only for religious purposes/occasions. But Vedas (written around 1500-900 BCE) in fact are a treasure house of knowledge and human experience—both secular and spiritual. Here you will get an idea about the power of Vedic Mathematics.

Vedic Number Representation

Vedic knowledge is in the form of slokas or poems in Sanskrit verse. A number was encoded using consonant groups of the Sanskrit alphabet, and vowels were provided as additional latitude to the author in poetic composition. The coding key is given as Kaadi nav, taadi nav, paadi panchak, yaadashtak ta ksha shunyam Translated as below·

VARNMALA

ka kha ga gha gna

cha chha ja jha inja

ta tha da dha na

pa pha ba bha ma

ya ra la va sha

sha sa ha chjha tra gna

- letter "ka" and the following eight letters
- letter "ta" and the following eight letters
- letter "pa" and the following four letters

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- letter "ya" and the following seven letters, and
- letter "ksha" for zero.

In other words,

- ka, ta, pa, ya = 1
- kha, tha, pha, ra = 2
- ga, da, ba, la = 3
- gha, dha, bha, va = 4
- gna, na, ma, scha = 5
- cha, ta, sha = 6
- chha, tha, sa = 7
- ja, da, ha = 8
- jha, dha = 9
- ksha = 0

Thus pa pa is 11, ma ra is 52. Words kapa, tapa , papa, and yapa all mean the same that is 11.

Tirthaji and the Rediscovery of Vedic Mathematics

The ancient system of Vedic Mathematics was rediscovered from the Sanskrit texts known as the Vedas, between 1911 and 1918 by Sri Bharati Krsna Tirthaji (1884-1960). At the beginning of the twentieth century, when there was a great interest in the Sanskrit texts in Europe, Bharati Krsna tells us some scholars ridiculed certain texts which were headed 'Ganita Sutras'— which means mathematics. They could find no mathematics in the translation and dismissed the texts as rubbish. Bharati Krsna, who was himself a scholar of Sanskrit, Mathematics, History and Philosophy, studied these texts and after lengthy and careful investigation was able to reconstruct the mathematics of the Vedas. According to his research all of mathematics is based on sixteen Sutras, or word-formulae.

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Bharati Krsna wrote sixteen volumes expounding the Vedic system but these were unaccountably lost and when the loss was confirmed in his final years he wrote a single book: Vedic Mathematics, currently available. It was published in 1965, five years after his death.

Development of Further Material

A copy of the book was brought to London a few years later and some English mathematicians (Kenneth Williams, Andrew Nicholas, Jeremy Pickles) took an interest in it. They extended the introductory material given in Bharati Krsna's book and gave many courses and talks in London. A book (now out of print), Introductory Lectures on Vedic Mathematics, was published in 1981. Between 1981 and 1987 Andrew Nicholas made four trips to India initially to find out what further was known about it. Following these journeys a renewed interest was taken by scholars and teachers in India. It seems that once they saw that some people in the West took Vedic Mathematics seriously they realised they had something special. St. James' School, then in Queensgate, London, and other schools began to teach the Vedic system, with notable success. Today Vedic Mathematics is taught widely in schools in India and a great deal of research is being done. Three further books appeared in 1984, the year of the centenary of the birth of Sri Bharati Krsna Tirthaji. These were published by The Vedic Mathematics Research Group.

Maharishi Schools

When Maharishi Mahesh Yogi began to explain the significance and marvellous qualities of Vedic Mathematics in 1988, Maharishi Schools around the world began to teach it. At the school in Skelmersdale, Lancashire a full course was written and trialled for 11 to 14 year old pupils, called The Cosmic Computer. (Maharishi had said that the Sutras of Vedic Mathematics are the software for the cosmic

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computer- the cosmic computer runs the entire universe on every level and in every detail). This course was published in March 1998.

Neither Vedic Nor Mathematics

As has been pointed out earlier on several occasions, the so-called 'Vedic Mathematics' is neither 'Vedic' nor can it be dignified by the name of mathematics. 'Vedic Mathematics', as is well-known, originated with a book of the same name by a former Shankracharya of Puri (the late Jagadguru Swami Shri Bharati Krishna Tirthaji Maharaj) published posthumously in 1965. The book assembled a set of tricks in elementary arithmetic and algebra to be applied in performing computations with numbers and polynomials. As is pointed out even in the foreword to the book by the General Editor, Dr. A. S. Agarwala, the aphorisms in Sanskrit to be found in the book have nothing to do with the Vedas. Nor are these aphorisms to be found in the genuine Vedic literature.

The term "Vedic mathematics" is therefore entirely misleading and factually incorrect. Further, it is clear from the notation used in the arithmetical tricks in the book that the methods used in this text have nothing to do with the arithmetical techniques of antiquity. Many of the Sanskrit aphorisms in the book are totally cryptic (ancient Indian mathematical writing was anything but cryptic) and often so generalize to be devoid of any specific mathematical meaning. There are several authoritative texts on the mathematics of Vedic times that could used in part to teach an authoritative and correct account of ancient Indian mathematics but this book clearly cannot be used for any such purpose. The teaching of mathematics involves both the teaching of the basic concepts of the subject as well as methods of mathematical computation. The so-called "Vedic mathematics" is entirely inadequate to this task considering that it is largely made up of tricks to do some elementary

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arithmetic computations. Many of these can be far more easily performed on a simple computer or even an advanced calculator.

The book "Vedic mathematics" essentially deals with arithmetic of the middle and high-school level. Its claims that "there is no part of mathematics, pure or applied, which is beyond their jurisdiction" is simply ridiculous. In an era when the content of mathematics teaching has to be carefully designed to keep pace with the general explosion of knowledge and the needs of other modern professions that use mathematical techniques, the imposition of ``Vedic mathematics" will be nothing short of calamitous.

India today has active and excellent schools of research and teaching in mathematics that are at the forefront of modern research in their discipline with some of them recognised as being among the best in the world in their fields of research. It is noteworthy that they have cherished the legacy of distinguished Indian mathematicians like Srinivasa Ramanujam, V. K. Patodi, S. Minakshisundaram, Harish Chandra, K. G. Ramanathan, Hansraj Gupta, Syamdas Mukhopadhyay, Ganesh Prasad, and many others including several living Indian mathematicians. But not one of these schools has lent an iota of legitimacy to `Vedic mathematics'. Nowhere in the world does any school system teach "Vedic mathematics" or any form of ancient mathematics for that matter as an adjunct to modern mathematical teaching. The bulk of such teaching belongs properly to the teaching of history and in particular the teaching of the history of the sciences.

We consider the imposition of `Vedic mathematics' by a Government agency, as the perpetration of a fraud on our children, condemning particularly those dependent on public education to a sub-standard mathematical education. Even if we assumed that those who sought to impose `Vedic mathematics' did so in good faith, it would have been

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appropriate that the NCERT seek the assistance of renowned Indian mathematicians to evaluate so-called "Vedic mathematics" before making it part of the National Curricular framework for School Education. Appallingly they have not done so. In this context we demand that the NCERT submit the proposal for the introduction of `Vedic mathematics in the school curriculum to recognized bodies of mathematical experts in India, in particular the National Board of Higher Mathematics (under the Dept. of Atomic Energy), and the Mathematics sections of the Indian Academy of Sciences and the Indian National Science Academy, for a thorough and critical examination. In the meanwhile no attempt should be made to thrust the subject into the school curriculum either through the centrally administered school system or by trying to impose it on the school systems of various States.

We are concerned that the essential thrust behind the campaign to introduce the so-called 'Vedic mathematics' has more to do with promoting a particular brand of religious majoritarianism and associated obscurantist ideas rather than any serious and meaningful development of mathematics teaching in India. We note that similar concerns have been expressed about other aspects too of the National Curricular Framework for School Education. We re-iterate our firm conviction that all teaching and pedagogy, not just the teaching of mathematics, must be founded on rational, scientific and secular principles.

Myths and Reality : On 'Vedic Mathematics'

School of Mathematics

We in India have good reasons to be proud of a rich heritage in science, philosophy and culture in general, coming to us down the ages. In mathematics, which is my own area of specialisation, the ancient Indians not only took great strides long before the Greek advent, which is a standard

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reference point in the Western historical perspective, but also enriched it for a long period making in particular some very fundamental contributions such as the place-value system for writing numbers as we have today, introduction of zero and so on. Further, the sustained development of mathematics in India in the post-Greek period was indirectly instrumental in the revival in Europe after "its dark ages".

Notwithstanding the enviable background, lack of adequate attention to academic pursuits over a prolonged period, occasioned by several factors, together with about two centuries of Macaulayan educational system, has unfortunately resulted, on the one hand, in a lack of awareness of our historical role in actual terms and, on the other, an empty sense of pride which is more of an emotional reaction to the colonial domination rather than an intellectual challenge. Together they provide a convenient ground for extremist and misguided elements in society to "reconstruct history" from nonexistent or concocted source material to whip up popular euphoria.

That this anti-intellectual endeavour is counter-productive in the long run and, more important, harmful to our image as a mature society, is either not recognised or ignored in favour of short-term considerations. Along with the obvious need to accelerate the process of creating an awareness of our past achievements, on the strength of authentic information, a more urgent need has also arisen to confront and expose such baseless constructs before it is too late. This is not merely a question of setting the record straight. The motivated versions have a way of corrupting the intellectual processes in society and weakening their very foundations in the long run, which needs to be prevented at all costs.

The so-called " Vedic mathematics" is a case in point. A book by that name written by Jagadguru Swami Shri Bharati Krishna Tirthaji Maharaja (Tirathji, 1965) is at the centre

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of this pursuit, which has now acquired wide following; Tirthaji was the Shankaracharya of Govardhan Math, Puri, from 1925 until he passed away in 1960. The book was published posthumously, but he had been carrying out a campaign on the theme for a long time, apparently for several decades, by means of lectures, blackboard demonstrations, classes and so on. It has been known from the beginning that there is no evidence of the contents of the book being of Vedic origin; the Foreword to the book by the General Editor, Dr. A. S. Agrawala, and an account of the genesis of the work written by Manjula Trivedi, a disciple of the swamiji, make this clear even before one gets to the text of the book. No one has come up with any positive evidence subsequently either.

There has, however, been a persistent propaganda that the material is from the Vedas. In the face of a false sense of national pride associated with it and the neglect, on the part of the knowledgeable, in countering the propaganda, even educated and well meaning people have tended to accept it uncritically. The vested interests have also involved politicians in the propaganda process to gain state support. Several leaders have lent support to the "Vedic mathematics" over the years, evidently in the belief of its being from ancient scriptures. In the current environment, when a label as ancient seems to carry considerable premium irrespective of its authenticity or merit, the purveyors would have it going easy.

Large sums have been spent both by the Government and several private agencies to support this "Vedic mathematics", while authentic Vedic studies continue to be neglected. People, especially children, are encouraged to learn and spread the contents of the book, largely on the baseless premise of their being from the Vedas. With missionary zeal several "devotees" of this cause have striven to take the "message" around the world; not surprisingly, they have even met with some success in the West, not

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unlike some of the gurus and yogis peddling their own versions of "Indian philosophy". Several people are also engaged in "research" in the new "Vedic mathematics."

To top it all, when in the early nineties the Uttar Pradesh Government introduced "Vedic mathematics" in school text books, the contents of the swamiji's book were treated as if they were genuinely from the Vedas; this also naturally seems to have led them to include a list of the swamiji's sutras on one of the opening pages (presumably for the students to learn them by heart and recite!) and to accord the swamiji a place of honour in the " brief history of Indian mathematics" described in the beginning of the textbook, together with a chart, which curiously has Srinivasa Ramanujan's as the only other name from the twentieth century!

For all their concern to inculcate a sense of national pride in children, those responsible for this have not cared for the simple fact that modern India has also produced several notable mathematicians and built a worthwhile edifice in mathematics (as also in many other areas). Harish Chandra's work is held in great esteem all over the world and several leading seats of learning of our times pride themselves in having members pursuing his ideas; (see, for instance, Langlands, 1993).

Even among those based in India, several like Syamdas Mukhopadhyay, Ganesh Prasad, B.N.Prasad, K. Anand Rau, T. Vijayaraghavan, S. S. Pillai, S. Minakshisundaram, Hansraj Gupta, K. G. Ramanathan, B. S. Madhava Rao, V. V. Narlikar, P. L. Bhatnagar and so on and also many living Indian mathematicians have carved a niche for themselves on the international mathematical scene (see Narasimhan, 1991). Ignoring all this while introducing the swamiji's name in the "brief history" would inevitably create a warped perspective in children's minds, favouring gimmickry rather than professional work.

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What does the swamiji's "Vedic mathematics" seek to do and what does it achieve? In his preface of the book, grandly titled" A Descriptive Prefatory Note on the astounding Wonders of Ancient Indian Vedic Mathematics," the swamiji tells us that he strove from his childhood to study the Vedas critically " to prove to ourselves (and to others) the correctness (or otherwise)" of the "derivational meaning" of "Veda" that the " Vedas should contain within themselves all the knowledge needed by the mankind relating not only to spiritual matters but also those usually described as purely 'secular', 'temporal' or 'worldly'; in other words, simply because of the meaning of the word 'Veda', everything that is worth knowing is expected to be contained in the vedas and the swamiji seeks to prove it to be the case!

It may be worthwhile to point out here that there would be room for starting such an enterprise with the word 'science'! He also describes how the " contemptuous or at best patronising " attitude of Orientalists, Indologists and so on strengthened his determination to unravel the too-long-hidden mysteries of philosophy and science contained in ancient India's Vedic lore, with the consequence that, "after eight years of concentrated contemplation in forest solitude, we were at long last able to recover the long lost keys which alone could unlock the portals thereof."

The mindset revealed in this can hardly be said to be suitable in scientific and objective inquiry or pursuit of knowledge, but perhaps one should not grudge it in someone from a totally different milieu, if the outcome is positive. One would have thought that with all the commitment and grit the author would have come up with at least a few new things which can be attributed to the Vedas, with solid evidence.

This would have made a worthwhile contribution to our understanding of our heritage. Instead, all said and done there is only the author's certificate that "we were agreeably

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astonished and intensely gratified to find that exceedingly though mathematical problems can be easily and readily solved with the help of these ultra-easy Vedic sutras (or mathematical aphorisms) contained in the Parishishta (the appendix portion) of the Atharva Veda in a few simple steps and by methods which can be conscientiously described as mere 'mental arithmetic' (paragraph 9 in the preface). That passing reference to the Atharva Veda is all that is ever said by way of source material for the contents. The sutras, incidentally, which appeared later scattered in the book, are short phrases of just about two to four words in Sanskrit, such as Ekadhikena Purvena or Anurupye Shunyam Anyat. (There are 16 of them and in addition there are 13 of what are called sub-sutras, similar in nature to the sutras).

The first key question, which would occur to anyone, is where are these sutras to be found in the Atharva Veda. One does not mean this as a rhetorical question. Considering that at the outset the author seemed set to send all doubting Thomases packing, the least one would expect is that he would point out where the sutras are, say in which part, stanza, page and so on, especially since it is not a small article that is being referred to. Not only has the author not cared to do so, but when Prof. K. S. Shukla, a renowned scholar of ancient Indian mathematics, met him in 1950, when the swamiji visited Lucknow to give a blackboard demonstration of his "Vedic mathematics", and requested him to point out the sutras in question in the Parishishta of the Atharva Veda, of which he even carried a copy (the standard version edited by G. M. Bolling and J. Von Negelein), the swamiji is said to have told him that the 16 sutra demonstrated by him were not in those Parishishtas and that "they occurred in his own Parishishta and not any other" (Shukla, 1980, or Shukla, 1991). What justification the swamiji thought he had for introducing an appendix in the Atharva Veda, the contents of which are nevertheless to be viewed as from the Veda, is anybody's guess. In any

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case, even such a Parishishta, written by the swamiji, does not exist in the form of a Sanskrit text.

Let us suppose for a moment that the author indeed found the sutras in some manuscript of the Atharva Veda, which he came across. Would he not then have preserved the manuscript? Would he not have shown at least to some people where the sutras are in the manuscript? Would he not have revealed to some cherished students how to look for sutras with such profound mathematical implications as he attributes to the sutras in question, in that or other manuscripts that may be found? While there is a specific mention in the write-up of Manjula Trivedi, in the beginning of the book, about some 16-volume manuscript written by the swamiji having been lost in 1956, there is no mention whatever (let alone any lamentation that would be due in such an event) either in her write-up nor in the swamiji's preface about any original manuscript having been lost. No one certainly has come forward with any information received from the swamiji with regard to the other questions above. It is to be noted that want of time could not be a factor in any of this, since the swamiji kindly informs us in the preface that "

Ever since (i.e. since several decades ago), we have been carrying on an incessant and strenuous campaign for the India-wide diffusion of all this scientific knowledge".

The only natural explanation is that there was no such manuscript. It has in fact been mentioned by Agrawala in his general editor's foreword to the book, and also by Manjula Trivedi in the short account of the genesis of the work, included in the book together with a biographical sketch of the swamiji, that the sutras do not appear in hitherto known Parishishtas. The general editor also notes that the style of language of the sutras "point to their discovery by Shri Swamiji himself" (emphasis added); the language style being contemporary can be confirmed independently from other

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Sanskrit scholars as well. The question why then the contents should be considered" Vedic" apparently did not bother the general editor, as he agreed with the author that "by definition" the Vedas should contain all knowledge (never mind whether found in the 20th century, or perhaps even later)! Manjula Trivedi, the disciple has of course no problem with the sutras not being found in the Vedas as she in fact says that they were actually reconstructed by her beloved " Gurudeva," on the basis of intuitive revelation from material scattered here and there in the Atharva Veda, after " assiduous research and 'Tapas' for about eight years in the forests surrounding Shringeri." Isn't that adequate to consider them to be "Vedic"? Well, one can hardly argue with the devout! There is a little problem as to why the Gurudeva himself did not say so (that the sutras were reconstructed) rather than referring to them as sutras contained in the Parishishta of the Atharva Veda, but we will have to let it pass. Anyway the fact remains that she was aware that they could not actually be located in what we lesser mortals consider to be the Atharva Veda.

The question of the source of the sutras is merely the first that would come to mind, and already on that there is such a muddle. Actually, even if the sutras were to be found, say in the Atharva Veda or some other ancient text, that still leaves open another fundamental question as to whether they mean or yield, in some cognisable way, what the author claims; in other words, we would still need to know whether such a source really contains the mathematics the swamiji deals with or merely the phrases, may be in some quite different context. It is interesting to consider the swamiji's sutras in this light. One of them, for instance, is Ekadhikena Purvena which literally just means" by one more than the previous one." In chapter I, the swamiji tells us that it is a sutra for finding the digits in the decimal expansion of numbers such as 1/19, and 1/29, where the denominator is a number with 9 in the unit's place; he goes on to give a page-

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long description of the procedure to be followed, whose only connection with the sutra is that it involves, in particular, repeatedly multiplying by one more than the previous one, namely 2, 3 and so on, respectively, the "previous one" being the number before the unit's place; the full procedure involves a lot more by way of arranging the digits which can in no way be read off from the phrase.

In Chapter II, we are told that the same sutra also means that to find the square of a number like 25 and 35, (with five in unit's place) multiply the number of tens by one more than itself and write 25 ahead of that; like 625, 1,225 and so on. The phrase Ekanyunena Purvena which means " by one less than the previous one" is however given to mean something which has neither to do with decimal expansions nor with squaring of numbers but concerns multiplying together two numbers, one of which has 9 in all places (like 99,999, so on.)! Allowing oneself such unlimited freedom of interpretation, one can also interpret the same three-word phrase to mean also many other things not only in mathematics but also in many other subjects such as physics, chemistry, biology, economics, sociology and politics. Consider, for instance, the following " meaning": the family size may be allowed to grow, at most, by one more than the previous one. In this we have the family-planning message of the 1960s; the "previous one" being the couple, the prescription is that they should have no more than three children.

Thus the lal trikon (red triangle) formula may be seen to be " from the Atharva Veda," thanks to the swamiji's novel technique (with just a bit of credit to yours faithfully). If you think the three children norm now outdated, there is no need to despair. One can get the two-children or even the one-child formula also from the same sutra; count only the man as the "previous one" (the woman is an outsider joining in marriage, isn't she) and in the growth of the family either count only the children or include also the wife, depending

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on what suits the desired formula! Another sutra is Yavadunam, which means "as much less;" a lifetime may not suffice to write down all the things such a phrase could "mean," in the spirit as above. There is even a sub-sutra, Vilokanam (observation) and that is supposed to mean various mathematical steps involving observation! In the same vein one can actually suggest a single sutra adequate not only for all of mathematics but many subjects:

It may be argued that there are, after all, ciphers which convey more information than meets the eye. But the meaning in those cases is either arrived at from the knowledge of the deciphering code or deduced in one or other way using various kinds of contexual information. Neither applies in the present case. The sutras in the swamiji's book are in reality mere names for various steps to be followed in various contexts; the steps themselves had to be known independently. In other words, the mathematical step is not arrived at by understanding or interpreting what are given as sutras; rather, sutras somewhat suggestive of the meaning of the steps are attached to them like names. It is like associating the 'sutra' VIBGYOR to the sequence of colours in rainbow (which make up the white light). Usage of words in Sanskrit, a language which the popular mind unquestioningly associates with the distant past(!), lend the contents a bit of antique finish!

An analysis of the mathematical contents of Tirthaji's book also shows that they cannot be from the Vedas. Though unfortunately there is considerable ignorance about the subject, mathematics from the Vedas is far from being an unexplored area. Painstaking efforts have been made for well over a century to study the original ancient texts from the point of view of understanding the extent of mathematical knowledge in ancient times. For instance, from the study of Vedic Samhitas and Brahamanas it has been noted that they had the system of counting progressing in multiples of 10 as we have today and that they considered remarkably

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large numbers, even up to 14 digits, unlike other civilizations of those times. From the Vedanga period there is in fact available a significant body of mathematical literature in the form of Shulvasutras, from the period between 800 BC and 500 BC, or perhaps even earlier, some of which contain expositions of various mathematical principles involved in construction of sacrificial 'vedi's needed in performing 'yajna's (see, for instance, Sen and Bag 1983).

Baudhyana Shulvasutra, the earliest of the extant Shulvasutras, already contains, for instance, what is currently known as Pythagoras' Theorem (Sen and Bag, 1983, page 78, 1.12). It is the earliest known explicit statement of the theorem in the general form (anywhere in the world) and precedes Pythagoras by at least a few hundred years. The texts also show a remarkable familiarity with many other facts from the so-called Euclidean Geometry and it is clear that considerable use was made of these, long before the Greeks formulated them. The work of George Thibaut in the last century and that of A.Burk around the turn of the century brought to the attention of the world the significance of the mathematics of the Shulvasutras. It has been followed up in this century by both foreign and Indian historians of mathematics. It is this kind of authentic work, and not some mumbo-jumbo that would highlight our rich heritage. I would strongly recommend to the reader to peruse the monograph, The Sulbasutras by S. N. Sen and A. K. Bag (Sen and Bag, 1983), containing the original sutras, their translation and a detailed commentary, which includes a survey of a number of earlier works on the subject. There are also several books on ancient Indian mathematics from the Vedic period.

The contents of the swamiji's book have practically nothing in common with what is known of the mathematics from the Vedic period or even with the subsequent rich tradition of mathematics in India until the advent of the modern era; incidentally, the descriptions of mathematical principles or procedures in ancient mathematical texts are

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quite explicit and not in terms of cryptic sutras. The very first chapter of the book (as also chapters XXVI to XXVIII) involves the notion of decimal fractions in an essential way. If the contents are to be Vedic, there would have had to be a good deal of familiarity with decimal fractions, even involving several digits, at that time. It turns out that while the Shulvasutras make extensive use of fractions in the usual form, nowhere is there any indication of fractions in decimal form. It is inconceivable that such an important notion would be left out, had it been known, from what are really like users manuals of those times, produced at different times over a prolonged period.

Not only the Shulvasutras and the earlier Vedic works, but even the works of mathematicians such as Aryabhatta, Brahmagupta and Bhaskara, are not found to contain any decimal fractions. Is it possible that none of them had access to some Vedic source that the swamiji could lay his hands on (and still not describe it specifically)? How far do we have to stretch our credulity? The fact is that the use of decimal fractions started only in the 16th century, propagated to a large extent by Francois Viete; the use of the decimal point (separating the integer and the fractional parts) itself, as a notation for the decimal representation, began only towards the end of the century and acquired popularity in the 17th century following their use in John Napier's logarithm tables (see, for instance, Boyer, 1968, page 334).

Similarly, in chapter XXII the swamiji claims to give " sutras relevant to successive differentiation, covering the theorems of Leibnitz, Maclaurin, Taylor, etc. and a lot of other material which is yet to be studied and decided on by the great mathematicians of the present-day Western world;" it should perhaps be mentioned before we proceed that the chapter does not really deal with anything of the sort that would even remotely justify such a grandiloquent announcement, but rather deals with differentiation as an operation on polynomials, which is a very special case

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reducing it all to elementary algebra devoid of the very soul of calculus, as taught even at the college level.

Given the context, we shall leave Leibnitz and company alone, but consider the notions of derivative and successive differentiation. Did the notions exist in the Vedic times? While certain elements preliminary to calculus have been found in the works of Bhaskara II from the 12th century and later Indian mathematicians in the pre-calculus era in international mathematics, such crystallised notions as the derivative or the integral were not known. Though a case may be made that the developments here would have led to the discovery of calculus in India, no historians of Indian mathematics would dream of proposing that they actually had such a notion as the derivative, let alone successive differentiation; the question here is not about performing the operation on polynomials, but of the concept. A similar comment applies with regard to integration, in chapter XXIV. It should also be borne in mind that if calculus were to be known in India in the early times, it would have been acquired by foreigners as well, long before it actually came to be discovered, as there was enough interaction between India and the outside world.

If this is not enough, in Chapter XXXIX we learn that analytic conics has an " important and predominating place for itself in the Vedic system of mathematics," and in Chapter XL we find a whole list of subjects such as dynamics, statics, hydrostatics, pneumatics and applied mathematics listed alongside such elementary things as subtractions, ratios, proportions and such money matters as interest and annuities (!), discounts (!) to which we are assured, without going into details, that the Vedic sutras can be applied. Need we comment any further on this?

The remaining chapters are mostly elementary in content, on account of which one does not see such marked incongruities in their respect. It has, however, been pointed

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out by Shukla that many of the topics considered in the book are alien to the pursuits of ancient Indian mathematicians, not only form the Vedic period but until much later (Shukla, 1989 or Shukla, 1991).

These include many such topics as factorisation of algebraic expressions, HCF (highest common factor) of algebraic expressions and various types of simultaneous equations. The contents of the book are akin to much later mathematics, mostly of the kind that appeared in school books of our times or those of the swamiji's youth, and it is unthinkable, in the absence of any pressing evidence, that they go back to the Vedic lore. The book really consists of a compilation of tricks in elementary arithmetic and algebra, to be applied in computations with numbers and polynomials. By a "trick" I do not mean a sleight of hand or something like that; in a general sense a trick is a method or procedure which involves observing and exploring some special features of a situation, which generally tend to be overlooked; for example, the trick described for finding the square of numbers like 15 and 25 with 5 in the unit's place makes crucial use of the fact of 5 being half of 10, the latter being the base in which the numbers are written. Some of the tricks given in the book are quite interesting and admittedly yield quicker solutions than by standard methods (though the comparison made in the book are facetious and misleading). They are of the kind that an intelligent hobbyist experimenting with numbers might be expected to come up with. The tricks are, however, based on well-understood mathematical principles and there is no mystery about them.

Of course to produce such a body of tricks, even using the well-known is still a non-trivial task and there is a serious question of how this came to be accomplished. It is sometimes suggested that Tirthaji himself might have invented the tricks. The fact that he had a M.A.degree in mathematics is notable in this context. It is also possible that he might have learnt some of the tricks from some

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elders during an early period in his life and developed on them during those "eight years of concentrated contemplation in forest solitude:" this would mean that they do involve a certain element of tradition, though not to the absurd extent that is claimed. These can, however, be viewed only as possibilities and it would not be easy to settle these details. But it is quite clear that the choice is only between alternatives involving only the recent times.

It may be recalled here that there have also been other instances of exposition and propagation of such faster methods of computation applicable in various special situations (without claims of their coming from ancient sources). Trachtenberg's Speed System (see Arther and McShane, 1965) and Lester Meyers' book, High-Speed Mathematics (Meyers, 1947) are some well-known examples of this. Trachtenberg had even set up an Institute in Germany to provide training in high-speed mathematics. While the swamiji's methods are independent of these, for the most part they are similar in spirit. One may wonder why such methods are not commonly adopted for practical purposes. One main point is that they turn out to be quicker only for certain special classes of examples. For a general example the amount of effort involved (for instance, the count of the individual operations needed to be performed with digits, in arriving at the final answer) is about the same as required by the standard methods; in the swamiji's book, this is often concealed by not writing some of the steps involved, viewing it as "mental arithmetic." Using such methods of fast arithmetic involves the ability or practice to recognise various patterns which would simplify the calculations. Without that, one would actually spend more time, in first trying to recognise patterns and then working by rote anyway, since in most cases it is not easy to find useful patterns.

People who in the course of their work have to do computations as they arise, rather than choose the figures suitably as in the demonstrations, would hardly find it convenient to carry them out by employing umpteen different

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ways depending on the particular case, as the methods of fast arithmetic involve. It is more convenient to follow the standard method, in which one has only to follow a set procedure to find the answer, even though in some cases this might take more time. Besides, equipment such as calculators and computers have made it unnecessary to tax one's mind with arithmetical computations. Incidentally, the suggestion that this "Vedic mathematics" of the Shankaracharya could lead to improvement in computers is totally fallacious, since the underlying mathematical principles involved in it were by no means unfamiliar in pro- fessional circles. One of the factors causing people not to pay due attention to the obvious questions about "Vedic mathematics" seems to be that they are overwhelmed by a sense of wonderment by the tricks. The swamiji tells us in the preface how "the educationists, the cream of the English educated section of the people including highest officials (e.g. the high court judges, the ministers etc.) and the general public as such were all highly impressed; nay thrilled, wonder-struck and flabbergasted!" at his demonstrations of the "Vedic mathematics." Sometimes one comes across reports about similar thrilling demonstrations by some of the present-day expositors of the subject. Though inevitably they have to be taken with a pinch of salt, I do not entirely doubt the truth of such reports. Since most people have had a difficult time with their arithmetic at school and even those who might have been fairly good would have lost touch, the very fact of someone doing some computations rather fast can make an impressive sight. This effect may be enhanced with well-chosen examples, where some quicker methods are applicable.

Even in the case of general examples where the method employed is not really more efficient than the standard one, the computations might appear to be fast, since the demonstrator would have a lot more practice than the people in the audience. An objective assessment of the methods from the point of view of overall use can only be made by comparing how many individual calculations are involved

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in working out various general examples, on an average, and in this respect the methods of fast arithmetic do not show any marked advantage which would offset the inconvenience indicated earlier. In any case, it would be irrational to let the element of sur- prise interfere in judging the issue of origin of " Vedic mathematics" or create a dreamy and false picture of its providing solutions to all kinds of problems.

It should also be borne in mind that the book really deals only with some middle and high school level mathematics; this is true despite what appear to be chapters dealing with some notions in calculus and coordinate geometry and the mention of a few, little more advanced topics, in the book. The swamiji's claim that "there is no part of mathematics, pure or applied, which is beyond their jurisdiction" is ludicrous. Mathematics actually means a lot more than arithmetic of numbers and algebra of polynomials; in fact multiplying big numbers together, which a lot of people take for mathematics, is hardly something a mathematician of today needs to engage himself in. The mathematics of today concerns a great variety of objects beyond the high school level, involving various kinds of abstract objects generalising numbers, shapes, geometries, measures and so on and several combinations of such structures, various kinds of operations, often involving infinitely many entities; this is not the case only about the frontiers of mathematics but a whole lot of it, including many topics applied in physics, engineering, medicine, finance and various other subjects. Despite all its pretentious verbiage page after page, the swamiji's book offers nothing worthwhile in advanced mathematics whether concretely or by way of insight.

Modern mathematics with its multitude of disciplines (group theory, topology, algebraic geometry, harmonic analysis, ergodic theory, combinatorial mathematics-to name just a few) would be a long way from the level of the swamiji's book. There are occasionally reports of some "re- searchers" applying the swamiji's "Vedic mathematics" to advanced

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problems such as Kepler's problem, but such work involves nothing more than tinkering superficially with the topic, in the manner of the swamiji's treatment of calculus, and offers nothing of interest to professionals in the area. Even at the school level "Vedic mathematics" deals only with a small part and, more importantly, there too it concerns itself with only one particular aspect, that of faster computation. One of the main aims of mathematics education even at the elementary level consists of developing familiarity with a variety of concepts and their significance. Not only does the approach of " Vedic mathematics" not contribute anything towards this crucial objective, but in fact might work to its detriment, because of the undue emphasis laid on faster computation. The swamiji's assertion "8 months (or 12 months) at an average rate of 2 or 3 hours per day should suffice for completing the whole course of mathematical studies on these Vedic lines instead of 15 or 20 years required according to the existing systems of the Indian and also foreign universities," is patently absurd and hopefully nobody takes it seriously, even among the activists in the area. It would work as a cruel joke if some people choose to make such a substitution in respect of their children.

It is often claimed that " Vedic mathematics" is well-appreciated in other countries, and even taught in some schools in UK etc.. In the normal course one would not have the means to examine such claims, especially since few details are generally supplied while making the claims. Thanks to certain special circumstances I came to know a few things about the St. James Independent School, London which I had seen quoted in this context. The School is run by the 'School of Economic Science' which is, according to a letter to me from Mr. James Glover, the Head of Mathematics at the School, "engaged in the practical study of Advaita philosophy". The people who run it have had substantial involvement with religious groups in India over a long period. Thus in essence their adopting " Vedic mathematics" is much like a school in India run by a religious group adopting

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it; that school being in London is beside the point. (It may be noted here that while privately run schools in India have limited freedom in choosing their curricula, it is not the case in England). It would be interesting to look into the background and motivation of other institutions about which similar claims are made. At any rate, adoption by institutions abroad is another propaganda feature, like being from ancient source, and should not sway us.

It is not the contention here that the con- tents of the book are not of any value. Indeed, some of the observations could be used in teaching in schools. They are entertaining and could to some extent enable children to enjoy mathematics. It would, however, be more appropriate to use them as aids in teaching the related concepts, rather than like a series of tricks of magic. Ultimately, it is the understanding that is more important than the transient excitement, By and large, however, such pedagogical application has limited scope and needs to be made with adequate caution, without being carried away by motivated propaganda.

It is shocking to see the extent to which vested interests and persons driven by misguided notions are able to exploit the urge for cultural self-assertion felt by the Indian psyche. One would hardly have imagined that a book which is transparently not from any ancient source or of any great mathematical significance would one day be passed off as a store- house of some ancient mathematical treasure. It is high time saner elements joined hands to educate people on the truth of this so-called Vedic mathematics and prevent the use of public money and energy on its propagation, beyond the limited extent that may be deserved, lest the intellectual and educational life in the country should get vitiated further and result in wrong attitudes to both history and mathematics, especially in the coming generation.

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4

The Sutra Era of Vedic Mathematics

The first glimpses of Mathematical activity in the Indian peninsula appear in the Sulva Sutras. The commonly accepted dates for such a dating are not without controversy. These conventional dates place the sutras, about 800 years before the common era. Here is one alternative chronology compiled by Lakshmikantham and Leela;

Proposed Skeleton of Indian Chronology 3374 BC

Event	Individual	Date
Birth		3374BCE
Birth	Veda Vyaasa	
Birth	Apasthambha	3200 BCE
Birth	Baudhayana	2765 BCE
Coronation of	Aryabhatta	1888-1807 BCE
Coronation of	Gautama Buddha	1554 BCE
Reign of	Chandragupta Maurya	1472 BCE
Coronation of	Ashoka Maurya	1294 - 1234 BCE
Pancha Siddhantas,	Kanishka	327 BCE
authored by	Chandragupta of Imperial	123 BCE
Reign of	Gupta dynasty	102 BCE to 78 CE
Birth of	Varahamihira	30 BCE
Era of	Vikramaaditya	78 CE
Writings	Brahmagupta	486 CE
	Salivahana	486 CE
	Bhaskara II	
	Siddhanta Siromani	

The politically correct date of the birth of Aryabhatta is 476 CE. Why the difference of 23 centuries. Apparently this

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resulted from a mistranslation of a Sanskrit word, where in fact one entire word has been replaced with another.

To quote Lakshmikantham and Leela, Page 27, Chapter 4, Chronological Aspects

Was Sir William indulging in subterfuge or was he woefully incompetent that he did not recognize the difference between shasti and shabdi

In the modern period of Bharat (India), Aryabhatta is the first famous mathematician and astronomer. In his book Aryabhatteeyam, Aryabhatta clearly provides his birth data. In the 10th stanza, he says that when 60 × 6 = 360 years elapsed in this Kali Yuga, he was 23 years old. The stanza of the sloka starts with "Shastyabdanam Shadbhiryada vyateetastra yascha yuga padah." "Shastyabdanam Shadbhi" means 60 × 6 = 360. While printing the manuscript, the word "Shadbhi" was altered to "Shasti", which implies 60 × 60 = 3600 years after Kali Era. As a result of this intentional arbitrary change, Aryabhatta's birth time was fixed as 476 CE Since in every genuine manuscript, we find the word "Shadbhi" and not the altered "Shasti", it is clear that Aryabhatta was 23 years old in 360 Kali Era or 2742 BCE. This implies that Aryabhatta was born in 337 Kali Era or 2765 BCE. and therefore could not have lived around 500 AD, as manufactured by the Indologists to fit their invented framework. Bhaskara I is the earliest known commentator of Aryabhatta's works. His exact time is not known except that he was in between Aryabhatta (2765 BCE) and Varahamihira (123 BCE).

Is this another case of Napoleon''s Dictum 'attribute not to malice that which can be explained away by sheer incompetence. You be the judge.

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The authors feel that such an error could not have crept in (remember it was a oral tradition) until the wording was changed after the British arrived in India and may have been done at the behest of Sir William.

The Indian Sulvasutras (also Pronounced Sulbasutras)

Cut the remaining parts (two-thirds of a strip) into eight equal strips and arrange them around the square we are constructing as in the diagram. We have now used all the parts of the second square but the new figure we have constructed is not quite a square having a small square corner missing. It is worth seeing what the side of this "not quite a square" is. It is

$$1 + 1/3 + 1/(3 \times 4)$$

which, of course, is the first three terms of the approximation. Now Datta argues in that to improve the "not quite a square" the Sulbasutra authors could have calculated how broad a strip one needs to cut off the left hand side and bottom to fill in the missing part which has area $\left(\frac{1}{12}\right)^2$. If x is the width one cuts off then

$$2 \times x \times \left(1 + \frac{1}{3} + \frac{1}{12}\right) = \left(\frac{1}{12}\right)^2.$$

This has the solution $x = 1/(3 \times 4 \times 34)$ which is approximately 0.002450980392. We now have a square the length of whose sides is

$$1 + \frac{1}{3} + \frac{1}{(3 \times 4)} - \frac{1}{(3 \times 4 \times 34)}.$$

which is exactly the approximation given by the Apastamba Sulbasutra.

Of course we have still made an approximation since the two strips of breadth x which we cut off overlapped by a

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square of side x in the bottom left hand corner. If we had taken this into account we would have obtained the equation

$$2\times x\times\left(1+\frac{1}{3}+\frac{1}{12}\right)-x^2=\left(\frac{1}{12}\right)^2$$

for x which leads to $x = \frac{17}{2} - v^2$ which is approximately equal to 0.002453105. Of course we cannot take this route since we have arrived back at a value for x which involves v2 which is the quantity we are trying to approximate!

In [4] Gupta gives a simpler way of obtaining the approximation for v2 than that given by Datta in [1]. He uses linear interpolation to obtain the first two terms, he then corrects the two terms so obtaining the third term, then correcting the three terms obtaining the fourth term. Although the method given by Gupta is simpler (and an interesting contribution) there is certainly something appealing in Datta's argument and somehow a feeling that this is in the spirit of the Sulbasutras.

Of course the method used by these mathematicians is very important to understanding the depth of mathematics being produced in India in the middle of the first millennium BC. If we follow the suggestion of some historians that the writers of the Sulbasutras were merely copying an approximation already known to the Babylonians then we might come to the conclusion that Indian mathematics of this period was far less advanced than if we follow Datta's suggestion.

Muhammad Ben Musa al-Khuwarizmi (circa 783-850)

Introduction

We are often told by western historians and scientists that Indians of the pre-Christian era were poor historians

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and even poorer at record keeping and hence that we know very little of the identities of the mathematicians and their contributions to the subject of mathematics. Our contention is precisely the opposite. Not only were the Indics superb record keepers , but they reported on the discoveries of their predecessors as well as contemporaries without the slightest sign of condescension or attempts to purloin the credit for themselves, a trait that appears to be singularly rare among those studying the field of Indology which seeks to obfuscate anything emanating out of India. In many cases the Vedic mathematicians were also the pre-eminent astronomers of their day. The problem lay not so much in the much bandied poor record keeping of the Indics, but in the fact that these records were either ignored or in certain egregious instances were in fact altered.

In Part I we furnished sufficient quotes to establish the proposition that throughout the ancient as well as the medieval eras Mediterranean, Arab and European savants went to great length to acknowledge the contribution of the Indics in various fields such as number theory, geometry ,astronomy, and medicine. It was only in the colonial era beginning with the discovery of Sir William Jones of the antiquity of Sanskrit, which had far reaching implications on the roots of their own civilization, that racial prejudice towards the Indics took on a dominant role and began to affect the quality as well as the accuracy of the scholarship in Europe. The roots of this prejudice and its progression over the 2 centuries of British rule in India are chronicled by Thomas Trautmann Everything, including the truth became subordinate to the paramount goal of maintaining a dominant role in India and Asia. We do not know for certain that there was deliberate falsification of records to support their versions of Indic history, but what we do know is that in certain key instances, the text has been altered to suit the pre-conceived notions of the Europeans on Indic history and that the general approach was to reject key

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pieces of data and dub them as being unreliable when it did not fit in with their overall paradigm of ancient Indic chronology. Before we get into specific acts of misdating, let us list chronologically the cast of characters who studied India under the general rubric of Indology.

List of Indologists Who Worked in the Area of Indology

Among these there are quite a few who neither harboured preconceived notions nor would they indulge in the dishonest act of altering documents. Typical of these were names like Playfair, Jacobi, Schopenhauer, Alain Danielou, Heinrich Zimmer and Joseph Campbell. But even with the best of intentions it is difficult to translate accurately from a language and culture which is alien to ones own. When that language is over 4000 years old, the difficulties are multiplied in manifold ways. When Europeans studied Sanskrit and the Vedas the paradigm they followed was that of studying insects in a jar. The science of studying insects is known as entomology. The insects for obvious reasons have little say in the matter. Such was the case also when they studied the civilization of the Indics.

The opinion of the Indics hardly mattered and they were rarely consulted and in many instances such as that of Max Mueller and Franz Bopp (max Mueller's professor) they had never set foot in India or conversed with a pundit. In just as many instances such as that of max Mueller they could not converse or chant a single sloka in Sanskrit much less understand one when it was chanted in front of them. But that did not stop them from claiming to be Sanskritists of the first rank. Neither did their dilettante status in Sanskrit stop Bopp and Sir William from deciding that there must have been an ancestral language (which they called Proto Indo European (PIE for short) spoken anywhere but in India. Now that I ponder on the reluctance of Max Mueller to visit India, the suspicion is overwhelming that the real reason he never wished to set foot in India was that he would

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thereby be spared the embarrassment of facing a real pundit in Sanskrit and have to then admit how shallow his knowledge of Sanskrit was.

Motivation for Present dating of Ancients

The original preoccupation of the European (to some extent true to this day) was to find the roots of his/her own language, which till the advent of Sanskrit was assumed to have been derived from Hebrew. The notion of a Hebraic origin was hardly very popular in Europe steeped as it was in anti Semitism. Therefore, when Sanskrit was first discovered, the notion that there was a race of noble Aryans who were their putative ancestors was then greeted with a great deal of enthusiasm once they had disposed of the prior suspicion that they were descended from the teeming millions of India.

The corollary to this proposition was that the denizens of the Indian subcontinent could not possibly have an antiquity greater than that of Greece or that of Pericles and the Shakespearean vision of the golden age of the Hellenic civilization. It was the conflict with his strict belief in the Creation theory postulated in the bible that led Sir William to lop of 1200 years from the Puranic history of India and to further postulate that the contemporary of Megasthenes , the Greek historian who visited India around 300 BCE was Chandragupta Maurya and not the Chandragupta of the Imperial Gupta dynasty.

We have already alluded to the postulate adhered to by almost all Indologists in the western world, that the Saraswati Sindhu civilization had little to do with the Vedic civilization in Part I of this series and we have described in great detail the efforts by the Colonial Power to undermine the Civilizational unity of India in the South Asia File. We wish to emphasize once again that the net result of all these efforts was to change the complexion of the debate. What was once a search for the roots of their own languages has

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now been transformed into, in their words, an obsession on the part of the Hindu right wing to prove that the Aryans were indigenous to India. This is undoubtedly a very astute strategy on their part since it takes the limelight of their own obsession to find a Urheimat for their group of languages and the fact that from the inception the postulates of dating Indic history have been political rather than academic in nature. The entire dating of the revisionist Indic history by the British and the Europeans has been a political enterprise right from the start. So, now when the argument is made that political considerations are driving the Hindu right wing in their opposition to theories such as the AIT, regardless of the truth of such an allegation, it ignores the glaring fact that it has always been so.

It is sad that a section of the Indic populace has internalized this revisionist view of Indic History propounded primarily by Europeans and it is important to remember that AIT is crucial to validate their racial view of civilizations and people and their sense of self esteem. They had to reconcile what was obviously a vast dependence on the contributions of the Semitic speaking people primarily along the Mediterranean Sea. In AIT they saw their opportunity to portray themselves as the progenitors of a vast Eurasian civilization without aligning themselves too closely to the brown skinned people of India. The conclusion is inescapable that while validating the AIT is not crucial to the pride of the Indic other than that it robbed him of his own authentic history, the debunking of AIT would have a devastating effect on the European weltanschauung of the roots of his own civilization. Indics in general for understandable reasons tend to be Indocentric and look to their own psyche to comprehend the nature of this paradigm. We, the Indics, would be much better served if we sought to understand the motivations and psyche of the European, at least in this instance or in other words to understand why there is such a constant emphasis on Eurocentrism in the European psyche.

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What do we expect to achieve by removing such misconceptions created essentially by a Eurocentric world. Every people need to have pride in their own traditions in order for there to be harmony, mutual respect and dignity among the peoples and civilizations of this world . Robbing them of their essential contributions is tantamount to breeding the seeds of racial superiority. What we are not seeking is retrospective privileges or an apology for such past misrepresentations, merely an acknowledgement that legitimate discoveries be attributed accurately to their rightful civilizations and that no single civilization or group of civilizations has a monopoly on the capability to advance the sum of human knowledge.

We will now discuss the anomalies in the assumptions made by European writers of Indic history and the evidence supporting our contention and see how devastating they were to a proper understanding and appreciation of Indic history and to a proper understanding of the Indic contribution to the sciences of antiquity.

Misdating of Aryabhatta the Elder

Aryabhatta is without doubt the Astronomer/ Mathematician non-pareil of the Post Vedic/Post Epic era in the historical narrative, especially so since his magnum opus The Aryabhattium, which packs a lot of information in the terse aphoristic style characteristic, of that era, has survived intact from the mists of a distant past when he first developed his thesis in 4 Chapters covering the subject of Mathematics and Astronomy. His work and the prior work in the Vedic area form an important sheet anchor for the entire chronology that follows, important by virtue of the fact that it attests to the state of the language prior to his contribution, and refers to the beginning of Kaliyuga as he reveals his own age. But first we list the main mathematicians during the period in question

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Some of the Vedic personalities that we will meet here are:

Yajnavalkya who wrote the Shatapatha Brahmana (as well as the Brihadaranyaka Upanishad) in which he describes the motion of the sun and the moon and advances a 95 year cycle to synchronize the motions of the sun and the moon

Lagadha who authored the Jyotisha Vedanga

Baudhayana the author of the Sulvasutra named after him

Apastambha the author of the Sulvasutra named after him

Katyayana the author of the Sulvasutra named after him

Panini the Grammarian for the Indo Europeans

Pingala Binary System of number representation:

Aryabhatta the astronomer laureate of ancient India

Varahamihira who synthesized the knowledge

The author of the Jaina treatises the Suryaprajnapati, Chandraprajnapati and the seventh section of Jambudvipaprajnapati

We will tackle first the misdating of Aryabhatta for the reasons stated above . Details of the life and contributions of this veritable genius are presented in the linked site. What concerns us most here is the episode relating to the misdating.

"Aryabhatta is the first famous mathematician and astronomer of Ancient India. In his book Aryabhatteeyam, Aryabhatta clearly provides his birth data. In the 10th stanza, he says that when 60 × 6 = 360 years elapsed in this Kali Yuga, he was 23 years old. The stanza of the sloka starts with "Shastyabdanam Shadbhiryada vyateetastra yascha yuga padah."

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"Shastyabdanam Shadbhi" means 60 × 6 = 360. While printing the manuscript, the word "Shadbhi" was altered to "Shasti", which implies 60 × 60 = 3600 years after Kali Era. As a result of this intentional arbitrary change, Aryabhatta's birth time was fixed as 476 AD. Since in every genuine manuscript, we find the word "Shadbhi" and not the altered "Shasti", it is clear that Aryabhatta was 23 years old in 360 Kali Era or 2742 BC. This implies that Aryabhatta was born in 337 Kali Era or 2765 BC, and therefore could not have lived around 500 AD, as manufactured by the Indologists to fit their invented framework.

Bhaskara I is the earliest known commentator of Aryabhatta's works. His exact time is not known except that he was in between Aryabhatta (2765 BC) and Varahamihira (123 BC)."

The implications are profound, if indeed this is the case. The zero is by then in widespread use and if he uses Classical Sanskrit then he antedates Panini.

"How the beginning of Kaliyuga is Linked with the Dates of Indian astronomers? The ancient Indian astronomers perhaps purposely linked the determination of their dates of birth, composition of their works; calculation of number of years elapsed, etc., based on two eras Kali and Saka. Therefore, without the significance of these two eras, the dates cannot be determined specifically.

Shastabdhanam shastardha vyatitastrashyam yugapadha |

Trayadhika vimsatirabdhastdheha mama janmanoatita | |

"When sixty times sixty years and three quarter yugas (of the current yuga) had elapsed, twenty three years had then passed since by birth" (K. S. Shukla).

"Now when sixty times sixty years and three quarter Yugas also have passed, twenty increased by three years have elapsed since my birth" (P. C. Sengupta).

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"I was born at the end of Kali 3600; I write this work when I am 23 years old i.e., at the end of Kali 3623"
"(T. S. Kuppanna Sastry11).

Here, though only Yuga is mentioned, Kaliyuga is implied and its starting of 3102 BCE is taken for reckoning purpose. Thus, the date of Aryabhatta is determined as follows:

The year of birth = 3600 - 3102 = 488/499 - 23 = 476 CE. This has been accepted by most of the scholars and generally considered as accepted date. Had the commencement year 3102 BCE is a myth or not astronomical one, the year of Aryabhatta cannot be historical date or could be determined like this using 3102 BCE.

Bhaskara I in his commentary to Aryabhatteeyam mentions as follows (Ch.I.verse.9):

Kalpadherabdhnirodhadhayam abdharashiritiritaha:

khagnyadhriramarkarasavasurandhrenadhavaha: te cangkkairapi 1986123730 |

"Since the beginning of the current Kalpa, the number of years elapsed is this: zero, three, seven, three, twelve, six, eight, nine, one (proceeding from right to left) years. The same (years) in figures are 1986123730".

"The time elapsed, in terms of years, since the commencement of the current kalpa is zero, three, seven, three, twelve, six, eight, nine, one (years written in figures) are 1986123730".

Aryabhatta gives the number of years elapsed since the beginning of the current yuga

= 6 Manus + 27 ¾ yugas

= 6 × 72 yugas + ¾ yugas

= 6 × 72 + 27 ¾) × 43,20,000 years

= (1866240000 + 119880000) years

= 198612000 years.

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From this, we can calculate the number of years elapsed since Bhaskara wrote his commentary

= 1986123730 -198612000 = 3730 years

= 3730 - 3102

= 628/629 CE.

Bhaskaracharya and others too imply Kaliyua/era in their works as revealed through commentators, as their dates are determined with the calculations reckoning the date of starting of Kaliyuga/era as 3102 BCE.

Mahabharata and Kaliyuga: The starting of Kaliyuga has been associated with the following events12:

The end of Mahabharat war.

The death of Sri Krishna.

The deluge, which made Dwaraka, submerged.

Coronation of Yudhistira.

The renouncement of Yuddhistira.

In Indian astronomical works including Tantras and Karanas, the word yuga has been taken as Kaliyuga for calculating, illustrative and explanatory purposes. In a Tantra, the epoch is the beginning of Kaliyuga or 3102 BCE. In a Karana, any convenient epoch is selected by the astronomer.

"The Saka year (when the civil days are required) added to 3179 gives the solar years elapsed since the beginning of the Kaliyuga" (Sisyadhivrddhida -Tantra13 - hereinafter mentioned as ST. - of Lalla.I.12).

Here, that the Saka year began 3179 after the beginning of the Kaliyuga is specifically mentioned. Moreover, in the calculation of days elapsed, solar years elapsed, Suddhi ertc., Kaliyuga is repeatedly mentioned and used for illustrations.

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"...the solar months elapsed since the beginning of the Kaliyuga multiplied by 22,26,389 and divided by 21,60,0000 give the corresponding lunar months" (ST.I.15).

The commentators Bhaskara I, Somesvara, Suryadeva Yajvan and others have pointed out the relation between Mahabharata and Kaliyuga.

Mahabharata and Kali Era: The usage of Kali era by the astronomers with the Mahabharata, that too, with Mahabharat war in particular, has been consistent. Many astronomers mention Kali era and Saka era together.

"Since the birth of Brahma up to the beginning of the Saka era, 8 ½ years (of Brahma), ½ month (of Brahma), 6 Manus of the (current) day (of Brahma), 27¾ yugas, and 3179 years of the (current) Kali era had gone by" (Vatesvara Siddhanta 14.I.10, K. S. Shukla).

Here, the number of years 3179 specifically mentioned is to obtain any year in terms of Saka, but it has been derived from the Kali era i.e, 3102/3101 + 78 = 3180/3179. As Vatesvara (c.880-960 CE) uses the notation, it is evident that even during the 9th century it had been very popular among the astronomers and established one. He also records his year of birth in that fashion as explained below.

Kali Era and Saka Era: After Aryabhatta, astronomers use the computation of years in Saka and as well as Kali Eras. The number of years to be reckoned in Saka with respect to Kali is given as 3179 and this is obtained by adding 78 to 3101/3102, thus, 3101 + 78 = 3179.

Thus, the Tantra directs: Navadhrirupagniyuttam mahibhujam shakendratnam gatavarshadaraham (I.4) meaning, "Add 3179 to the Saka years elapsed, the Kali years elapsed are obtained". Thus, it is evident that such method of reckoning of years in Saka Era related to Kali Era and vice versa had been in vogue before 6th century.

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Vateswara says: "When 802 years had elapsed since the commencement of the Saka era, my birth took place; and when 24 years had passed since my birth, this Siddhanta was written by me by the grace of the heavenly bodies" (I.21).

Thus, the year of birth = 802 + 78 = 880 CE and that of his work = 880 + 24 = 904 CE.

The astronomers use certain Sakas as illustrative examples in their works. For example Mallikarjuna Suri and Candesvara, an astronomer of Mitthila use 1100 and 1107 Sakas for illustrating rules.

Mahabharata and 3102 BCE: The above discussion about Kaliyuga and Kali era amply proves its astronomical importance in time reckoning. It also points to the well established date of such reckoning starting with 3102 BCE and its connection with Mahabharata. The Saka Era has also been associated with it as starting 3179 after the starting of Kali Era. The date 3102/3101 BCE is very often used by the epigraphists, numismatists, archaeologists, historians, astronomers and others, but, assert that Kali Era/Yuga is a myth! Therefore, the application of 3102 BCE to determine and calculate other dates should be explained properly, because, the modern scholars use the same date, but even condemn and criticize it unwittingly at many places.

Bhaskara mentions the names of Latadeva, Nisanku and Panduranga Svami as disciples of Aryabhatta.

So the question is which version of the sloka in Aryabhattium is the correct one

Is it this one?

Shastyabdanam Shadbhiryada vyateetastra yascha yuga padah."

"Shastyabdanam Shadbhi" means 60 × 6 = 360. Which places his birth at 2765 BCE (360 -23 - 3102)

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Or this one

Shastabdhanam shastardha vyatitastrashyam yugapadha |

Trayadhika vimsatirabdhastdheha mama janmanoatita | |

Shastabdhanam shastardha means 60 × 60 = 3600

Which places his birth at (3600 - 23 - 3102) = 475 CE

The resulting shift in the date of Aryabhatta of 3240 years, makes him roughly 3 times higher in chronology , and has profound consequences for the Indic contributions relative to those of Babylonian mathematicians

The Vedanga Period (Vyakarana, Jyotisha, Chandas, Kalpa Sutra)

The Vedanga (IAST vedanga, "member of the Veda") are six auxiliary disciplines for the understanding and tradition of the Vedas.

1. Shiksha (siksa): phonetics and phonology (sandhi)
2. Chandas (chandas): meter Pingala
3. Vyakarana (vyakarana): grammar Panini
4. Nirukta (nirukta): etymology Yaska
5. Jyotisha (jyotisa): astrology Lagadha
6. Kalpa (kalpa): ritual Apastambha, Baudhayana, Katyayana , Manava

The Vedangas are first mentioned in the Mundaka Upanishad as topics to be observed by students of the Vedas. Later, they developed into independent disciplines, each with its own corpus of Sutras.

During this period we will consider the timeline of the following contributors:

Panini - Vyakarana (Grammar, Place value system) was one of the many all time great savants that India has produced

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in such abundance. The scope and scale of his vast contributions to language, grammar, computing science, and place value system is mind numbing at a time when scripts were in an embryonic stage of development.

It is no hyperbole to say that there was no study of grammar as a codified set of rules until the west discovered Pannini's Ashtadhyayi

Pingala

Yaska

Apastambha

Baudhayana

Katyayana

Ashvalayana

Timeline According to Max Mueller

Chandas Rg Veda	1200 to 1000 BCE
Mantras later Vedas	1000 to 800 BCE
Brahmanas	800 to 600 BCE
Sutras	600 to 200 BCE

Timeline According to Keith

Taittiriya Samhita	500 BCE
Baudhayana	400 BCE
Ashvalayana	350 BCE
Sankhayana	350 BCE
Yaska	300 BCE
Apastambha	300 BCE
Pratisakhya	300 BCE
Panini	250 BCE
Katyayana	800to 600 BCE

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Assumptions

The Sulvasutras precede the developments in Babylon and Egypt and must therefore date at least to 2000 BCE.

The Rigveda must have been fully composed prior to 2000 BCE because of the drying up of the Saraswati River prior to 1900 BCE.

The Vedangas indicate a knowledge of the use of zero and the place value system.

Seidenberg on the significance of the Sulvasutras.

Seidenberg, A. On the volume of a sphere. Arch. Hist. Exact Sci. 39 (1988), no. 2, 97--119. (Reviewer: K.-B. Gundlach.) SC: 01A20 (01A15 01A17 01A25 01A32), MR: 89j:01012.

Abraham Seidenberg argues that there is a common source for Pythagorean and Chinese (or Chinese-like) mathematics. He suggests that Old-Babylonian mathematics is a derivative of a more ancient mathematics having a much clearer geometric component (p. 104), and is "in some respects ... is derivative of a Chinese-like mathematics" (p. 109). Van der Waerden holds a similar view on this, and tells us that the mathematics of the Chiu Chang Suan Shu represents the common source more faithfully than the Babylonian does. Seidenberg believes that the common source is most similar to the Sulvasutras. He discusses how questions of the sphere and the circle were treated by the Greeks, Chinese, Egyptians, and to a lesser extent Indians. He discusses the some similarities and differences in the work on the sphere in Greece (Archimedes, with a very brief account of the application of his Method), and in Chinese (first in the Chiu Chang Suan Shu, improved by Liu Hui or perhaps Tsu Ch'ung-Chih, and then further improved by the Tsu Ch'ung-Chih's son Tsu Keng-Chih). He believes that the problem of the volume of a sphere goes back to the common source, to the first part of the second millennium BC, or

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earlier. An interesting and related topic is the topic of the equality of the proportionality constants pi that occur in the formulas for the area and circumference of a circle. Seidenberg examines the Moscow Papyrus, Chinese sources, and an Old-Babylonian text and finds that this fact seemed to be recognized in all three groups. He argues that the Egyptian, Babylonian, and Chinese approaches to the volume of a truncated pyramid may have derived from the same common source. He believe that the common source also used infinitesimal, Cavalieri-type, arguments as well. It is interesting as well that Heron, who as Seidenberg notes is sometimes considered to be continuing the Babylonian tradition, gives the formula 1/2(s+p)p+1/14(1/2s)2 for the area of a segment of a circle with chord s and height (sagita, arrow) p (with an Archimedean value of 22/7 for pi), and "that the 'ancients' took [the area as] 1/2(s+p)p and even conjectured that they did so because they took pi = 3." The paper is also interesting in that he discusses the development of some of his ideas from his early papers in the 60s until much later (the paper was received soon before his death). Closely related topics: The Sphere, The Circle, The Pythagoreans, China, The Chiu Chang Suan Shu (Nine Chapters on the Mathematical Art), Sumerians and Babylonians, The Sulvasutras, Archimedes, Archimedes' Method, The Moscow Mathematical Papyrus, and Heron.

Mathews, Jerold. A Neolithic oral tradition for the van der Waerden/Seidenberg origin of mathematics. Arch. Hist. Exact Sci. 34 (1985), no. 3, 193--220. (Reviewer: M. Folkerts.) SC: 01A10 (01A25), MR: 88b:01005.

Abraham Seidenberg advanced a theory that mathematics arose from a common origin, and that some the mathematics was preserved by an oral tradition, and very likely a religious tradition, perhaps one like the one seen in the Indian Sulvasutras. Van der Waerden's book Geometry and Algebra in Ancient Civilizations takes a similar views, and in fact van der Waerden credits Seidenberg for making

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him look at the history of mathematics a new way. As Mathews notes, the Chinese Chiu Chang Suan Shu is very important in van der Waerden's work. Mathews relies heavily on this work as well to "give a small, coherent, and basic core of geometry concerning rectangles and their parts, ..., which may serve as what van der Waerden has called an 'oral tradition current in the Neolithic age.'" He states the he hoped "to give this hypothesized ancient core some credence through its relation to the Chiu Chang and its explanatory power. After giving a thorough discussion of this geometry, he then carefully analyzes the ninth chapter of the Chiu Chang in terms of this core. He is able to find a strong match, though his conclusions on one of the problems (Problem 20) are not consistent with those of some other researchers, who find in problem 20 instead suggestions of something like Horner's method. A very interesting article. Hopefully future papers will discuss how well the author's geometry agrees with the ancient geometry of other cultures. As he notes, "Until I can thoroughly test his conjecture on, say, the Babylonian corpus, I can argue for the merits of my conjecture only on such grounds as the simplicity of explanation it allows, or its congruence with received results or figures." Closely related topics: The Neolithic Era, Religion, Geometry, The Chiu Chang Suan Shu (Nine Chapters on the Mathematical Art), and Abraham Seidenberg.

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5

Vedic Maths, a Forgotten Science

It is being taught in some of the most prestigious institutions in England and Europe. NASA scientists applied its principles in the area of artificial intelligence. And yet, in the country of its birth it languishes as a forgotten science.

Vedic mathematics, which simplifies arithmetic and algebraic operations, has increasingly found acceptance the world over. Experts suggest that it could be a handy tool for those who need to solve mathematical problems faster by the day.

What is Vedic Mathematics?

It is an ancient technique, which simplifies multiplication, divisibility, complex numbers, squaring, cubing, square and cube roots. Even recurring decimals and auxiliary fractions can be handled by Vedic mathematics.

Who Brought Vedic Maths to Limelight?

The subject was revived largely due to the efforts of Jagadguru Swami Bharathikrishna Tirthaji of Govardhan Peeth, Puri Jaganath (1884-1960). Having researched the subject for years, even his efforts would have gone in vain but for the enterprise of some disciples who took down notes during his last days. That resulted in the book, Vedic

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Mathematics, in the 1960s. These are now available in a book called "VEDIC MATHEMATICS".

What is the Basis of Vedic Mathematics?

The basis of Vedic mathematics, are the 16 sutras, which attribute a set of qualities to a number or a group of numbers. The ancient Hindu scientists (Rishis) of Bharat in 16 Sutras (Phrases) and 120 words laid down simple steps for solving all mathematical problems in easy to follow 2 or 3 steps.

Vedic Mental or one or two line methods can be used effectively for solving divisions, reciprocals, factorisation, HCF, squares and square roots, cubes and cube roots, algebraic equations, multiple simultaneous equations, quadratic equations, cubic equations, bi-quadratic equations, higher degree equations, differential calculus, Partial fractions, Integrations, Pythogorus theoram, Apollonius Theoram, Analytical Conics and so on.

What is the Speciality of Vedic Mathematics?

Vedic scholars did not use figures for big numbers in their numerical notation. Instead, they preferred to use the Sanskrit alphabets, with each alphabet constituting a number. Several mantras, in fact, denote numbers; that includes the famed Gayatri mantra, which adds to 108 when decoded.

Is it Useful Today?

Given the initial training in modern maths in today's schools, students will be able to comprehend the logic of Vedic mathematics after they have reached the 8th standard. It will be of interest to every one but more so to younger students keen to make their mark in competitive entrance exams.

India's past could well help them make it in today's world.

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It is amazing how with the help of 16 Sutras and 16 sub-sutras, the Vedic seers were able to mentally calculate complex mathematical problems.

1. Why did the Vedic Seers need Vedic Mathematics or Mental Mathematics?

   The Vedic Seers were highly ritualistic in practice. All the four Vedas namely Rig Veda, Yajur Veda, Sama Veda and Atharva Veda, consist of Samhitas, Brahmanas, Aranyakas and Upanishads. Of these four, the first three namely Samhitas, Brahmanas and Aranyakas contain several thousand Mantras or Hymns, Ritual practices and their interpretations. The Vedic Seers were very particular about the time of doing the rituals. Hence they needed a very scientific and accurate calendar and time measurements. Hence we have the adage IST which once meant Indian Standard Time. But in recent times due to our laziness and carelessness, it has come to mean Indian Stretchable Time.

   The Vedic culture was that of Yajna and the Vedic Purohits or priests were also very particular about the shape and size of the Yajna Kund. For this they needed a very highly developed Geometry and Trigonometry.

2. Where is Vedic Mathematics found?

   Vedic Mathematics forms part of Jyotish Shastra which is one of the six parts of Vedangas. The Jyotish Shastra or Astronomy is made up of three parts called Skandas. A Skanda means the big branch of a tree shooting out of the trunk

Some of the Sutras (Phrases) used in Vedic Mathematics:

1. Ekadhikena Purvena (One More than the Previous) is useful in solving Special Multiplications like 25 × 25, 95 × 95, 105 × 105 etc.

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Special Divisons like 1 divided by 19, 29, 39, ... 199 etc.

2. Eka Nunena Purvena (One less than the Previous) is useful in solving Special Multiplications like 777 × 999, 123456789 × 999 999 999.
3. Urdhva Tiryak bhyam (Vertically and Crosswise) is useful in General Multiplication of any number by any number.
4. Paravartya Yojayet (P-64) (Transpose and Apply) is useful in solving Algebraic factors and divisions of some numbers etc.
5. Anurupyena (P-87) (Proportionately).
6. Adhyam-Adhyena, Antyam-antyena (P-87) (first by the first and the last by the last) are useful in solving Quadratics.
7. Lopana-stapana-bhyam (P-90) (by (alternate) Elimination and Retention) is useful in factorizing long and harder quadratics.
8. Gunita-Samuchhayah Samuchhaya-gunitah which means "The product of the sum of the coefficients in the factors is equal to the sum of the coefficients in the product" is a Sub-sutra of immense utility for the purpose of verifying the correctness of our answers in multiplications, divisions and factorisations.
9. Sunyam Samya samuccaye P-107 (when Samuccaya is the same, that Samuccaya is zero) Samuccaya is a technical term which has several meanings. This is useful in solving many complex factors and equations.

1st Example—1 Divided by 19, 29, 39, ... 129, etc.

1. To Divide 1 by numbers ending in 9 like 1 divided by 19, 29, 39, .,.... 119, etc.

Some of these numbers like 19, 29, 59 are prime numbers and so cannot be factorised and division becomes all the

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more difficult and runs into many pages in the present conventional method and the chances of making mistakes are many.

The Vedic Solution is obtained by applying the Sutra (theorem) Ekadhikena Purvena which when translated means "One more than the Previous".

Take for example 1 divided by 19. In the divisor 19, the previous is 1 and the factor is obtained by adding 1 to it which is 2. Similarly when we have to divide by 29, 39, ... 119 the factors shall be 3,4,... 12 respectively. (Add 1 to the previous term in the divisor). After this divide 1 by the factor in a typical Vedic way and the answer is obtained in 1 step. Thus:

1 divided by 19 = 0.0 5 2 6 3 1 5 7 8 9 4 7 3 6 8 4 2 1

1 divided by 29 = 0.0 3 4 4 8 2 7 5 8 6 2 0 6 8 9 6 5
5 1 7 2 4 1 3 7 9 3 1

2nd Example—Square of Numbers ending in 5

Squares of 25, 35, 45, 85, 95, 105, 195, etc. can be worked out mentally.

Again the Sutra used here is Ekadhikena Purvena which means, "One more than the previous."

The last term is always 5 and the Previous terms are 2, 3, 4, 8, 9, 10, 19 etc. and we have to add 1 to them. Square of the last term 5 is always 25.

Thus the Square of 25 is 2 × 3/25 = 625

the Square of 35 is 3 × 4/25 = 1225

the Square of 45 is 4 × 5/25 = 2025

the Square of 85 is 8 × 9/25 = 7225

the Square of 95, 105, 195 can be obtained in the same way.

Use the above formula to find the products of 23 multiplied by 27; 44 multiplied by 46; 192 multiplied by 198 and so on.

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3rd Example—Multiplier-digits Consist Entirely of Nines

The Sutra (theorem) used here is Ekanyunena Purvena, sound as if it were the converse of Ekadhik Sutra i.e "one less"

777 multiplied by 999 = 776,223

(776 is one less than multiplicand 777 and 223 is the compliment of 776 from 9)

120 35 79 multiplied by 999 99 99 = 120 35 78, 879 64 21

1234 5678 09 multiplied 9999 9999 99
= 1234 5678 08 8765 4321 91

Such multiplications come up in advanced astronomy.

4th Example—General Multiplication of any Number by any Number

The Sutra used is Urdhva-Tiryagbhyam which means, "Vertically and crosswise" (P-39)

To multiply 12 by 13 mentally multiply

1 by 1, 3 by 1 and 2 by 1 and finally 2 by 3 and write the answer as 1 56

To multiply 37 by 42 mentally multiply

3 by 4, 3 by 2 and 7 by 4 and finally 7 by 2 and the answer is 12/34/14 = 1554

To multiply 1021 by 2103 mentally multiply as follows

1 by2 1by1+0by2 1by0+2by2+1by0 1by3 + 0by0 + 2by1 + 1by2 0by3 + 2by0 + 1by1 2by3 + 1by0 1by3 = 2147163

Multiplying 8 7 2 6 5 by 3 2 1 1 7
gives 2 8 0 2 6 9 0 0 0 5

5th Example—Algebraic Divisions

Divide ($12x^2$ -8x -32) by (X -2) using Urdhva-Tiryak, (Vertically and Crosswise)

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Just by observation we can say the quotient must be (12X+k)

We also know that -8x = kx - 24x Hence k = 16 cross-check -2k = -32 third term

Divide (X^3 + $7X^2$ +6X +5) by (X-2)

X^3 divided by X gives X2 which is therefore the 1st term of the equation.

X^2 multiplied by -2 gives -2x2, But we have $7x^2$ in the Dividend. This means we have to get $9X^2$ more. This must result from the multiplication of X by 9X. Hence the 2nd term of the divisor must be 9X. Hence the quotient Q= X^2+ 9x + k...

For the third term we already have 6X = kX - 18X.

Hence k = 24

To find the Remainder R: 2k+5 = 53

6th Example—Division

Sutra: Paravartya Yojayet "Transpose and Apply" (P-63)

Divide ($12x^2$ - 8x - 32) by (x - 2), factor is +2

(X - 2)/2 $12x^2$ - 8x -32

+24x + 32

12x + 16 R = 0

Divide $7x^2$ + 5x + 3 by x - 1 Q = 7x + 12 R = 15

Divide $7x^2$ + 5x + 3 by x + 1 Q = 7x -2 R = 5

Divide x^3 + $7x^2$ + 6x + 5 by x - 2 Q = x^2 + 9x + 24 R = 53

Divide x^4 - x^3 + x^2 + 3x +5 by x^2 - x - 1 Factors are x + 1

Q = x^2 + 0x + 2 R = 5x + 7

Divide $6x^4$ + $13x^3$ + $39x^2$ + 37x + 45 by X^2 - 2x - 9

Factors 2x + 9 Q = $6x^2$ + 25x + 143 R = 548x + 1332

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Divide $x^4 + x^2 + 1$ by $x^2 - x + 1$ (add $0x^3$ & $0x$)

$X^4 + 0x^3 + x^2 + 0x + 1$

Factors are: x - 1

$Q = x^2 + x + 1$ R = 0

Divide 1 2 3 4 by 1 1 2

Factors -1 -2 Q = 11 R = 2

Divide 1 3 4 5 6 by 1 1 2 3 Q = 12 R = - 20

The Reminder cannot be negative.

Hence Q = 11 R = 1 1 0 3

Divide 1 3 9 0 5 by 1 1 3 Q = 1 24 R = -107

Q = 123 R = 6

7th Example—Factorisations of Quadratics

Sutras: Anurupyena (P-84) "Proportionately."

Adhyam-Adhyena, Antyam-antyena (P-84) "first by the first and the last by the last"

8th Example—Verifying Correctness of Answers

A Sub-sutra of immense utility for the purpose of verifying the correctness of our answers in multiplications, divisions and factorisations:

Gunita-Samuchhayah Samuchhaya-gunitah (P-86) means

"The product of the sum of the coefficients in the factors is equal to the sum of the coefficients in the product"

S_c of the product = Product of the S_c in the factors

For example $(x+7)(x+9) = (x^2 + 16x + 63)$

$(1+7)(1+9) = (1 + 16 + 63) = 80$

or$(x+1)(x+2)(x+3) = (x^3 + 6x^2 + 11x + 6)$

$(1+1)(1+2)(1+3) = (1 + 6 + 11 + 6) = 24$

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9th Example—Factorisations of Harder Quadratics

Lopana-stapana-bhyam (P-87) "by (alternate) Elimination and Retention"

It is very difficult to factorise the long quadratic

$(2x^2 + 6y^2 + 3z^2 + 7xy + 11yz + 7zx)$

But "Lopana-Sthapana" removes the difficulty. Eliminate z by putting z = 0.

Hence the given expression

$E = 2x^2 + 6y^2 + 7xy = (x+2y)(2x+3y)$

Similarly, if y=0, then $E = 2x^2 + 3z^2 + 7zx = (x+3z)(2x+z)$

Hence $E = (x + 2y+3z)(2x+3y+z)$

Factorise $2x^2 + 2y^2 + 5xy + 2x - 5y - 12 = (x+3)(2x-4)$ and $(2y+3)(y-4)$

Hence, $E = (x+2y+3)(2x+y-4)$

*This "Lopana-sthapana" method (of alternate elimination and retention) will be found highly useful in HCF, in Solid Geometry and in Co-ordinate Geometry of the straight line, the Hyperbola, the conjugate Hyperbola, the Asymptotes etc.

10th Example—Factorisations of Harder Quadratics-Special Cases

Sunyam Samya samuccaye P-107 (when Samuccaya is the same, that Samuccaya is zero) Samuccaya is a technical term which has several meanings.

First Meaning: It is a term which occurs as a common factor in all the terms concerned

Thus $12x + 3x = 4x + 5x$ x is common.

Hence $x = 0$

Or $9(x+1) = 7(x+1)$ here $(x+1)$ is common.

Hence $x + 1 = 0$

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Second Meaning: Here Samuccaya means "the product of the independent terms"

Thus, (x +7) (x +9) = (x +3) (x + 21)

Here 7 x9 = 3 x 21. Therefore x = 0

Third Meaning: Samuccaya thirdly means the sum of the Denominators of two fractions having the same numerical numerator

Thus, 1/(2x -1) + 1/(3x -1) = 0.

Hence 5x - 2 =0 or x = 2/5

Fourth Meaning: Here Samuccaya means combination (or TOTAL).

In this context it is used in different contexts. These are

If the sum of the Numerators and the sum of the Denominators be the same, then that sum = 0

(2x +9)/ (2x +7) = (2x +7)/ (2x +9)

N1 + N2 = D1 + D2 = 2x + 9 + 2x + 7 = 0

Hence 4x + 16 = 0.

Hence x = -4

Note: If there is a numerical factor in the algebraic sum, that factor should be removed.

(3x +4)/ (6x +7) = (x +1)/ (2x +3)

Here N1 +N2 = 4x +5; D1 +D2 = 8x + 10;

4x +5 =0 x= -5/4

Fifth Meaning: Here Samuccaya means TOTAL i.e Addition & subtraction

Thus, (3x +4)/ (6x +7) = (5x +6)/ (2x +3)

Here N1+N2 = D1 + D2 = 8x + 10 = 0.

Hence x = - 5/4

D1 - D2 = N2 - N1 = 2x + 2 = 0 x = -1

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Sixth Meaning: Here Samuccaya means TOTAL; used in Harder equations

Thus, 1/ (x-7) + 1/(x-9) = 1/(x-6) + 1/(x-10)

Vedic Sutra says, (other elements being equal), the sum-total of the denominators on LHS and the total on the RHS are the same, then the total is zero.

Here, D1 + D2 = D3 + D4 = 2x-16 = 0.

Hence x = 8

Examples 1/(x+7) + 1/(x+9) = 1/(x+6) + 1/(x+10) x = - 8

1/(x-7) + 1(x+9) = 1/(x+11) + 1/(x-9) x = - 1

1/(x-8) + 1/(x-9) = 1/(x-5) + 1/(x-12) x = 8-1/2

1/(x-b) - 1/(x-b-d) = 1/(x-c+d) - 1/(x-c) x = 1/2(b+c)

Special Types of seeming Cubics

(x- 3)3 + (x -9)3 = 2(x -6)3

current method is very lengthy, but Vedic method says,

(x-3) + (x-9) = 2x - 12 Hence x = 6

$(x-149)^3 + (x-51)^3 = 2(x-100)^3$

Hence 2x-200 =0 & x = 100

$(x+a+b-c)^3 + (x+b+c-a)^3 = 2(x+b)^3$ x = -b

Sixteen Simple Mathematical Sutras (Phrases) From the Vedas

Sixteen Sutras and Their Corollaries

Sutras or Formulae	*Sub-Sutras or Corollaries*
Ekadhikena Purvena	Anurupyena
(One more than the previous) P-2	(Proportionately) P-20, 87
Division eg. 1/19	Multiplications
Nikhilam Navatascaram Dashtah	Sishyate Seshasanjnah
(All from 9 last from 10) P-13	
Multiplication eg 9927 X 9999	

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Urdhva-tiryaghyam P- (Vertically and crosswise)	Adhyam-adhyena antyam-antyena P-87 (First by the first and last by the last)
Paravartya Yojayet P- (Transpose and apply)	Kevalaih Saptakam Gunyat
Sunyam Samya samuccaye P-107 (when Samuccaya is the same, that Samuccaya is zero)	Veshtanam
(Anurupye) Sunyamanyat	Yavadunam Tavadunam
Sankalana-vyavakalana	Yavadunam Tavadunikrtya Vargamca Yojayet
Purna purnabhyam	Antyayor Dasakepi (also for two numbers whose last digits together total 10)
Calana-Kalanabhyam	Antyayoreva
Yavadunam	Samuccayagunitah
Vyashti samashti	Lopana Sthapanabhyam "by (alternate) Elimination and Retention" very useful in HCF, Solid Geometry, Coordinate Geometry,
Seshanyankena caramena	Vikolanam
Sopantya dvayamantyam	Gunita samuccayah Samucchaya gunitah P-89 *13
Ekanyunena purvena (one less than the previous) P-35 one of the multiplier-digits is all 9	
Gunita Samuccayah P-	
16. Gunaka Samuccayah	

Notes and Quiz

To many Indians Vedic and Sanskrit Mantras are relevant only for religious occasions like weddings, Grahapraveshas and Upanayanams. Some who are aware

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of Upanishads and Bhagwad Gita, are fascinated by the depth of their philosophical teachings. But Vedas in fact are a store-house of knowledge - both secular and spiritual eg. Ayur Veda is a treatise on medicine and health. Dhanur Veda on war techniques and martial arts, Stapathya Veda on buildings and constructions, Gandharva Veda on Music and art, and Artha Shastra on political systems and economics.

- According to scholars like Max Muller, Vedas are the oldest texts of mankind. They are the treasure house of both secular and spiritual knowledge.
- Schaupenhopher, the German scholar said,

  "The Upanishads are the solace of my life and they shall be the solace of my death too".

  * Upanishads are the Philosophical portions of the Vedas.
- Once a great Vedic scholar Mandana Mishra remarked, "If the Vedas are true nothing shall happen to me" and he jumped from a tall building. To every one's surprise he landed safely but one of his legs was bruised little bit. He questioned the Ved Bhagwan why he was hurt and he got the answer why he was punished.

Can you guess the answer?

- Once a German Indologist remarked "India's contribution in Science in zero"

Do you think he was: (a) right (b) wrong (c) ill-informed (d) biased against India

- If you were to keep one rice grain on the 1st square of the chess board, 2 grains on the 2nd square, 4 grains on the 3rd, 8 on the 4th, 16 on the 5th, 32 on the 6th and so on, ... how many kilograms of rice would you need to fill all the squares in a chess board?

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Vedic Sciences

As it is the Rigveda, which pointed to the Cosmic Law first, I feel aggrieved to note that all the ancient wisdom that the vedic seers discovered has been attributed to other nations. The discovery of the Natural Law is credited to the Stoics of Greece who existed circa 300 B.C. whereas the Rigveda was composed at least in the 3500 B.C., if not earlier. This is the price of political weakness, and the Hindus have still not stopped paying for it. Until they learn to be strong, the erosion of their cultural greatness will not only continue, but is likely to accelerate its pace.

'Earliest writing' found

The first known examples of writing may have been unearthed at an archaeological dig in Pakistan.

So-called 'plant-like' and 'trident-shaped' markings have been found on fragments of pottery dating back 5500 years.

They were found at a site called Harappa in the region where the great Harappan or Indus civilisation flourished four and a half thousand years ago.

Harappa was originally a small settlement in 3500 BC but by 2600 BC it had developed into a major urban centre.

The earliest known writing was etched onto jars before and after firing. Experts believe they may have indicated the contents of the jar or be signs associated with a deity.

According to Dr. Richard Meadow of Harvard University, the director of the Harappa Archaeological Research Project, these primitive inscriptions found on pottery may pre-date all other known writing.

Last year it was suggested that the oldest writing might have come from Egypt.

Clay tablets containing primitive words were uncovered in southern Egypt at the tomb of a king named Scorpion.

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They were carbon-dated to 3300-3200 BC. This is about the same time, or slightly earlier, to the primitive writing developed by the Sumerians of he Mesopotamian civilisation around 3100 BC.

"It's a big question as to if we can call what we have found true writing," he told BBC News Online, "but we have found symbols that have similarities to what became Indus script.

"One of our research aims is to find more examples of these ancient symbols and follow them as they changed and became a writing system," he added.

One major problem in determining what the symbols mean is that no one understands the Indus language. It was unique and is now dead.

Dr. Meadow points out that nothing similar to the 'Rosetta stone' exists for the Harappan text. The Rosetta stone contained Egyptian hieroglyphics and a Greek translation and thus helped early language researchers decipher the meaning.

The Harappan language died out and did not form the basis of other languages.

"So probably we will never know what the symbols mean," Dr. Meadow told BBC News Online from Harappa.

What historians know of the Harappan civilisation makes them unique. Their society did not like great differences between social classes or the display of wealth by rulers. They did not leave behind large monuments or rich graves.

They appear to be a peaceful people who displayed their art in smaller works of stone.

Their society seems to have petered out. Around 1900 BC Harappa and other urban centres started to decline as people left them to move east to what is now India and the Ganges.

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This discovery will add to the debate about the origins of the written word.

It probably suggests that writing developed independently in at least three places - Egypt, Mesopotamia and Harappa between 3500 BC and 3100 BC.

About Multiple Methods of Multiplication

Why look at other ways to multiply when our usual algorithm works just fine? First, this provides alternatives for kids who struggle with the usual algorithm. Second, thinking about how these other algorithms work makes us think more deeply about numbers and operations. Third, it's kind of fun to figure out how people from other times and places using other numeration systems computed. The Egyptian method for multiplication, in particular, makes us appreciate the ease of calculation in the number system which comes to us from the Hindu Vedic tradition.

Ancient Egyptian Multiplication

The Egyptian numeration system was radically different from the one we use today. Both that system and our current system represent numerals with powers of 10. Beyond that there is not much similarity. Our current system has a different symbol to represent each number from 0 through 9, and the placement of the symbol in the numeral determines whether that symbol represents the number of ones, or tens, or hundreds, etc. The Egyptian system had a different symbol for each power of ten, and the number of copies of the symbol indicated how many of that power of ten was in the numeral. This is an awkard numeration system in which to create an easy algorithm for multiplication. Multiplication is just repeated addition of a number. The multiplication algorithm which the Egyptians came up was remarkably efficient use of repeated addition to get around the limitations of a number system without place value. It is probably the first application of binary numbers.

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Vedic Mathematics

The history of mathematics in India is ancient. The Hindu Vedic tradition is an oral tradition of knowledge passed down in short verses, dating to before the invention of paper. The Vedas encompass a broad spectrum of knowledge, including the sutras (verses) pertaining to mathematics. In the early 20th century Swami Shri Bharati Krishna Tirthaji Maharaja claimed to have rediscovered a collection of 16 ancient mathematical sutras from the Vedas and published it in a book called Vedic Mathematics. Historians do not agree on whether or not these were truly part of the Vedic tradition. If these sutras date back to the Vedic era they were certainly part of an oral rather than a written tradition.

However, they are a novel and useful approach to computation: they are flexible in application and easy to remember. They can often be applied in algebraic contexts as well as in simple arithmetic.. "Vertically and Crosswise" bridges the gap between arithmetic and algebra: the algorithm is very similar to the standard algorithm used in the US and also is similar to the "FOIL" (first, outer, inner, last) rule used for multiplying binomials in algebra.

The oldest surviving math texts from India include a text treating math from a geometric perspective, the Sulbasutra written by Baudhayana around the 9th century BC, and the Aryabhatiya by Aryabhatta, written around the 5th century AD. If the Vedic Sutras collected by Bharati Krishna have a historical source, they are probably older than either of these texts. They are in any case an interesting alternative approach.

Our number system is derived from the Hindu numeration system which evolved from Vedic mathematics. The concept of zero and the Hindu-Arabic numerals found their way to Europe in the 12th century with Fibonacci, 5 centuries after the work by Hindu mathematician

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Brahmagupta. It is the place value system with zero which makes it possible for us to use our familiar multiplication algorithm and spares us from coping with the Egyptian algorithm or the abacus.

Why have a picture of a lamp on this display? The lamp is a symbol for the Hindu festival of Diwali, an Indian festival of lights. One of many interpretations of Diwali is a celebration of the light of knowledge scattering the darkness of ignorance.

Napier's Bones

Napier's main contribution to mathematics, logarithms, arose from his desire to take the tedium out of computation. The difficult computations of his time were multiplication, division, and square and cube roots. Accurate computation was important for applications in navigation--the stars were the global positioning system of Renaissance Europe, and locating a position on the globe required careful computations involving trigonometry. The bones, of course, were an aid to quick and accurate multiplication. Using the bones for multiplication is similar to using the lattice, or Gelosia, method of multiplication.

Materials Preparation for this Module

The materials needed for this are a set of Napier's Bones and handouts for Egyptian multiplication. The Printable material includes a set of paper Napier's Bones which are intended to be cut out and glued onto 3/4 inch craft sticks (extra large popsicle sticks). I like to have three copies of each "bone".

Running the Activity

The multiplication problems are not provided on this website. The person in charge of the display can make up multiplication problems and encourage the visitors to the display to try one of the other methods of working the

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problem. He or she will of course want to practice doing a few problems using each of these methods before kids show up. If there are difficulties producing problems, the problems given in Notes to the Volunteer can be used.

Mathevedics Frequently Asked Questions

Q: What is Mathevedics?

Answer: Mathevedics is a unique mathematics teaching method. This method has elements of Vedic mathematics and mathematical methods specially developed by well qualified and experienced individuals. The method developed this way provides a unique tool to our students to "fly through" mathematics an alternate way which is extremely fast and a lot of fun! Our method is patent pending and it is illegal to copy either our method or material even in parts. We constantly work on developing and renewing our material. Our purpose is to provide a tool to students to look at Mathematics in a more fundamental and intelligent way whether it is arithmetic, algebra, geometry or calculus.

Q: What is Vedic Mathematics?

Answer: Vedic Mathematics is a form of mathematics developed in ancient times by mathematicians and thinkers from India. These methods are very succinct and if used with necessary explanations provide solutions to complicated mathematical problems in very simplified forms. It has been in use in India on a small scale all along. People trying to get an edge over others in speed of solving problems have been successfully using it in India .

Q: Is Mathevedics Vedic Mathematics?

Answer: Mathevedics uses elements of Vedic Mathematics. It is not completely Vedic mathematics. Our course material has been developed with a lot of effort by very qualified individuals using many original ideas and new methods in addition to Vedic Mathematics.

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Q: Does it apply to school mathematics?

Answer: Yes, it applies to School mathematics. We have ensured to bridge the gap between our alternate methods and traditional school methods in such a way that transition appears to be very seamless and natural to our students and it enhances their ability to appreciate mathematics.

Q: Will it confuse children as it applies to the same curriculum as School?

Answer: From our experience, it does not confuse children. We demonstrate a seamless transition between the school method and the alternate fast method to students. This re-enforces what is learnt in the school in addition to be able to look at the school method in a more basic way. As a result the student develops mathematical intelligence.

Q: Does it also cover algebra, geometry, trigonometry and calculus?

Answer: Yes, it very much covers algebra, geometry, trigonometry and calculus. Some people not totally familiar with Vedic techniques may think that Vedic techniques applied only to arithmetic. It is not true. Vedic Methods are originally in the form of verses which provide much generalized solutions to problems. Arithmetic is only an easy implementation and verification of Vedic methods. Mathevedics methods use elements of Vedic methods in many cases and are not limited by any limitation of Vedic Mathematics.

Q: How is Mathevedics different from Vedic Math?

Answer: Vedic Mathematics provides generalized solutions to many mathematical problems. Without proper interpretation of the generalized solutions as stated in Vedic Mathematics, it may appear to be very limited in scope and in most cases useless as many people have been thinking.

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Mathevedics methods use elements of Vedic Mathematics which other wise seem not very useful to many people. In addition to Vedic elements our method uses modern mathematics and mostly specially developed mathematical methods. Mathevedics combines these in such a way that the gap between these are bridged. Most of our method is developed from fundamentals by our team of experts. It provides a broader outlook of mathematics and is intended to ingrain the fundamentals of mathematics into the students' minds and provide the student a source of mathematical intelligence.

Q: How different is this method from Kumon, Sylvan Learning, Score, etc.?

Answer: We teach only Mathematics. We do not focus on learning by repetition. We focus on improving intelligence by teaching fundamentals and alternate methods. Our purpose is not limited to improving performance in the school or tests (which of course is one of our objectives), but we focus on providing a broader outlook resulting in improved mathematical intelligence. We provide one on one attention to our students. No other institute or learning centre does that.

Q: Some schools allow students to use calculator in the class? What is the need of Mathevedics?

Answer: Yes, it is true that some schools allow calculator in the class or even tests. One of the objectives of Mathevedics is providing a method to do complicated arithmetic calculations fast and in many cases in minds. Therefore we do provide a tool to reduce dependence on calculators.

However our main objective is to make the students mathematically intelligent by providing alternate methods and giving a broader perspective of standard problems. In addition to making the student intelligent, this boosts their confidence. The methods apply to Algebra, geometry and calculus as well.

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Therefore in order to go that extra mile in Mathematics the students will still need us even though they have calculators.

Calculators are not permitted in SAT etc. SAT in many cases is a career determining test and it is important to perform well there.

Q: Some one is selling Vedic Mathematics material for less than $200 in India without restrictions. Why would anyone try to get a Mathevedics Franchise if he can do the same thing for less money?

Answer: We are aware of Vedic Mathematics teaching resources in India and also at other places in the world. We only use elements of Vedic Mathematics. Most of our materiai is developed from fundamentals by a team of experts independent of Vedic Mathematics to provide to the student a source of Mathematical intelligence. We are confident that no one else can provide to the student what we provide and the objective of no Vedic Mathematics teaching centre or resource is to improve mathematical intelligence. Our method is patent pending and our material is copyrighted. Mathevedics will take legal action against anyone trying to steal the concept or the teaching material even in part.

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Bibliography

A.L. Basham: *The Wonder That Was India*, Rupa & Co., Calcutta, 1967.

Aaboe, Asger: *Episodes from the Early History of Mathematics*, New York, 1964.

Alfred, C.: *Introduction to Prakrit*, Calcutta, Baptist Mission Press, 1917.

Amma, S: *Geometry in Ancient and Medieval India*, Motilal Banarsidas, 1979.

Archibald, R.C.: *The Rhind Mathematical Papyrus,* Oberlin, Ohio, 1927.

Armstrong, G.: *Yoruba Numerals*, London, Oxford University Press, 1962.

Arnett, S.: *The Predynastic Origin of Egyptian Hieroglyphs*, Washington, University Press of America, 1982.

Ascher, Marcia: *Ethnomathematics*, Pacific Grove, CA: Brooks/Cole, 1991.

Bag, A. K.: *Mathematics in Ancient and Medieval India*, Varanasi: Chaukhambha Orientalia, 1979.

Barnard, F. P.: *The Casting-Counter and the Counting Board. A Chapter in the History of Numismatics and Early Arithmetic*, Oxford, Oxford University Press, 1916.

Barrow, D.: *Pi in the Sky*, Oxford, Clarendon Press, 1992.

Basham, A. L.: *The Wonder That Was India*, Calcutta, Rupa & Co., 1967.

Beckmann, Petr: *A History of Pi*, New York, St. Martin's Press, 1974.

Bedient, J. D.: *The Historical Roots of Elementary Mathematics*, New York, Dover, 1988.

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Bell, Eric T.: *The Magic of Numbers*, New York, McGraw-Hill, 1946.

Berlin, Brent: *Tzeltal Numeral Classifiers*, The Hague, Mouton, 1968.

Bharati Krishna Tirthaji Majaraja.: *Vedic Mathematics*. Delhi, Motilal Banarsidass, 1988.

Bidyaranya, Swami: *History of Hindu Mathematics*, Bombay, Asia Pub. House, 1962.

Bischoff, Bernard: *Latin Palaeography*, Cambridge, Cambridge University Press, 1990.

Bonfante, Larissa: *Etruscan*. Berkeley, University of California Press, 1990.

Brainerd, Charles J.: *The Origins of the Number Concept*, New York, Praeger, 1979.

Brunschwig, Jacques : *Greek Thought: A Guide to Classical Knowledge*, Cambridge, Harvard University Press, 2000.

Bunt, L. N. H.: *The Historical Roots of Elementary Mathematics*, New York, Dover, 1988.

Charles, P.: *The Numeration, Calendar Systems and Astronomical knowledge of the Mayas*, Cambridge, Cambridge University Press, 1910.

Charles, P.: *The Numeration, Calendar Systems and Astronomical knowledge of the Mayas*, Cambridge, Cambridge University Press, 1910.

Chattopadhyaya, D. P.: *Mathematics, Astronomy, and Biology in Indian Tradition: Some Conceptual Preliminaries*, New Delhi, Project of History of Indian Science, Philosophy, and Culture, 1995.

Clawson, Calvin: *The Mathematical Traveller*, New York, Plenum Press, 1994.

Closs, P.: *Native American Mathematics*, Austin, University of Texas Press, 1986.

Conant, L.: *The Number Concept*, New York, MacMillan, 1896.

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Crossley, N: *The Emergence of Number*, Singapore, World Scientific, 1987.

Crump, Thomas: *The Anthropology of Numbers*, Cambridge, Cambridge University Press, 1990.

Dantzig, Tobias: *Number, the Language of Science*, London, George Allen and Unwin, 1940.

David, F.: *Cross-Cultural Studies in Cognition and Mathematics*, New York, Academic, 1983.

De Villiers, Melius: *The Numeral-Words: Their Origin, Meaning, History and Lesson*, London, Witherby, 1923.

Dehaene, Stanislas: *The Number Sense*, New York, Oxford University Press, 1997.

Dilke, Wentworth: *Mathematics and Measurement*, Berkeley, CA: U. of Cal. Press, 1987.

Dirk, J.: *A Concise History of Mathematics*, New York, Dover, 1987.

Eric, S.: *Maya Hieroglyphic Writing: An Introduction*, Norman, University of Oklahoma Press, 1971.

Eves, Howard: *An Introduction to the History of Mathematics*, New York, Rinehart and Company Inc., 1953.

Flegg, Graham: *Numbers: Their History and Meaning*, London, André Deutsch, 1983.

Fowler, David: *The Mathematics of Plato's Academy*, Oxford, Clarendon, 1999.

Francis, Llewellyn: *Hieratic Papyri from Kahun and Gurob*, London, Bernard Quaritch, 1898.

Fuson, K.C.: *Children's Counting and Concepts of Number*, New York, Springer-Verlag, 1988.

Ganeri, Anita: *The Story of Numbers and Counting*, New York, Oxford University Press, 1996.

Geoffrey E.R. Lloyd:. *Greek Thought: A Guide to Classical Knowledge*, Cambridge, Harvard University Press, 2000.

Gokhale, Laxman: *Indian Numerals*, Poona, Deccan College, 1966.

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Griffith, Llewellyn: *Hieratic Papyri from Kahun and Gurob*, London, Bernard Quaritch, 1898.

Grover, Sudarshan: *History of Development of Mathematics in India*, Delhi, Atma Ram & Sons, 1994.

Haldane, Sanderson: *Science and Indian Culture*, Calcutta, New Age Publishers, 1965.

Halliwell, Orchard: *Rara Mathematica*, London, J.W. Parker, 1839.

Herbert Meschkowski: *Ways of Thought of Great Mathematicians*, Holden-Day Inc., San Francisco, 1964.

Hogben, Lancelot: *Mathematics for the Million*, New York, W. W. Norton, 1937.

Howard Eves: *An Introduction to the History of Mathematics*, Rinehart and Company Inc., New York, 1953.

Ifrah, Georges: *From One to Zero: A Universal History of Numbers*, New York, Viking Penguin, 1985.

Jagadguru Swami Shri Bharati Krishna Tirthaji Maharaja: *Vedic Mathematics*, Motilal Banarsidass, Delhi, 1988.

Jaggi, O. P.: *Science and Technology in Medieval India*, New Delhi, Atma Ram & Sons, 1977.

James, Orchard: *Rara Mathematica*, London, J.W. Parker, 1839.

John, Burdon Sanderson: *Science and Indian Culture*, Calcutta, New Age Publishers, 1965.

John, D.: *Pi in the Sky*, Oxford, Clarendon Press, 1992.

John, N: *The Emergence of Number*, Singapore, World Scientific, 1987.

Jones, D.: *The Internal Representation of Number: Analogue or Digital?*, Oxford, Clarendon, 1987.

Kline, Morris: *Mathematical Thought from Ancient to Modern Times*, New York, Oxford U. Press, 1990.

Kuppuram, G & K. Kumudamani: *History of Science and Technology in India*, Delhi, Sundeep Prakashan, 1990.

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Digitized by Arya Samaj Foundation Chennai and eGangotri

Lancy, F.: *Cross-Cultural Studies in Cognition and Mathematics*, New York, Academic, 1983.

Levi, L.: *The Number Concept*, New York, MacMillan, 1896.

Manning, H.P. and R C Archibald: *The Rhind Mathematical Papyrus,* Oberlin, Ohio, 1927.

Marcus, Joyce: *Mesoamerican Writing Systems*, Princeton, Princeton University Press, 1992.

Marshack, Alexander: *The Roots of Civilization*, New York, McGraw-Hill, 1972.

Martzloff, Jean-Claude: *A History of Chinese Mathematics*, New York, Springer, 1997.

McLeish, John: *The Story of Numbers*, New York, Fawcett Columbine, 1991.

Menninger, Karl: *Number-Words and Number-Symbols.* London, MIT, 1969.

Michael, P.: *Native American Mathematics*, Austin, University of Texas Press, 1986.

Murthy, T. S. Bhanu: *A Modern Introduction to Ancient Indian Mathematics*, Wiley Eastern Ltd., New Delhi, 1992.

Neugebauer, Otto : *Mathematical Cuneiform Texts*, New Haven, Yale University Press, 1945.

Neugebauer, Otto: *The Exact Sciences in Antiquity*, Princeton, Princeton University Press, 1952.

North, John: *The Norton History of Astronomy and Cosmology*, New York, W.W.Norton, 1995.

Oswald Ashton Wentworth: *Mathematics and Measurement*, Berkeley, CA: U. of Cal. Press, 1987.

Pandit, M. D.: *Mathematics as known to the Vedic Samhitas*, Delhi, Sri Satguru Publications, 1993.

Pappas, Theoni: *The Magic of Mathematics*, San Carlos, Wide World Publishing / Tetra, 1994.

Parpola, Asko: *Deciphering the Indus Script*, Cambridge, Cambridge University Press, 1994.

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Peet, T. Eric: *The Rhind Mathematical Papyrus*, London, British Museum, 1923.

Piaget, Jean: *The Child's Conception of Number*, New York, W.W. Norton, 1965.

Pullan, J.M: *The History of the Abacus*, New York, Praeger, 1968.

Rao, S. Balaachandra: *Indian Mathematics and Astronomy*, Bangalore, Jnana Deep Publications, 1994.

Restivo, Sal: *The Social Relations of Physics, Mysticism and Mathematics*, Dordrecht, Kluwer, 1983.

Robert, G.: *Yoruba Numerals*, London, Oxford University Press, 1962.

Robins, G. and C. Shute: *The Rhind Mathematical Papyrus*, London, British Museum, 1987.

Russell, Bertrand: *The Principles of Mathematics*, Cambridge, Cambridge University Press, 1903.

Sachs, A.: *Mathematical Cuneiform Texts*, New Haven, Yale University Press, 1945.

Salomon, Richard: *Indian Epigraphy*, New York, Oxford University Press, 1998.

Saraswarti, A.: *Geometry in Ancient and Medieval India*, Motilal Banarsidas, 1979.

Saraswati Amma: *Geometry in Ancient and Medieval India*, Motilal Banarsidas, 1979.

Sarton, George: *The Study of the History of Mathematics*, Cambridge, Cambridge University Press, 1936.

Satterthwaite, Jr.: *Concepts and Structures of Maya Calendrical Arithmetics*, Philadelphia, University of Pennsylvania, 1947.

Satterthwaite, Jr.: *Concepts and Structures of Maya Calendrical Arithmetics*, Philadelphia, University of Pennsylvania, 1947.

Schaaf, W.L.: *Mathematics, Our Great Heritage*, New York, Harper and Row, 1948.

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Schimmel, Annemarie: *The Mystery of Numbers*, New York, Oxford University Press, 1993.

Scriba, C.J.: *The Concept of Number*, Zurich, Mannheim, 1968.

Seidenberg, A.: *The Zero in the Mayan Numerical Notation*, Austin, University of Texas Press, 1986.

Shebhana, Laxman: *Indian Numerals*, Poona, Deccan College, 1966.

Smeltzer, Donald: *Man and Number*, New York, Emerson, 1959.

Smith, D. E.: *History of Mathematics*, New York, Dover Publications Inc., 1951.

Srinivasiengar, C. N.: *The History of Ancient Indian Mathematics*. Calcutta, World Press, 1967.

Struik, J.: *A Concise History of Mathematics*, New York, Dover, 1987.

Thompson, Sir J. : *Maya Hieroglyphic Writing: An Introduction*, Norman, University of Oklahoma Press, 1971.

Todd, R.R.: *The Internal Representation of Number: Analogue or Digital?*, Oxford, Clarendon, 1987.

Urton, Gary: *The Social Life of Numbers*, Austin, University of Texas Press, 1997.

Volodarsky, A.I.: *Mathematics in Ancient India*, Tokyo, Science Council of Japan, 1974.

William, S.: *The Predynastic Origin of Egyptian Hieroglyphs*, Washington, University Press of America, 1982.

Woolner, C.: *Introduction to Prakrit*, Calcutta, Baptist Mission Press, 1917.

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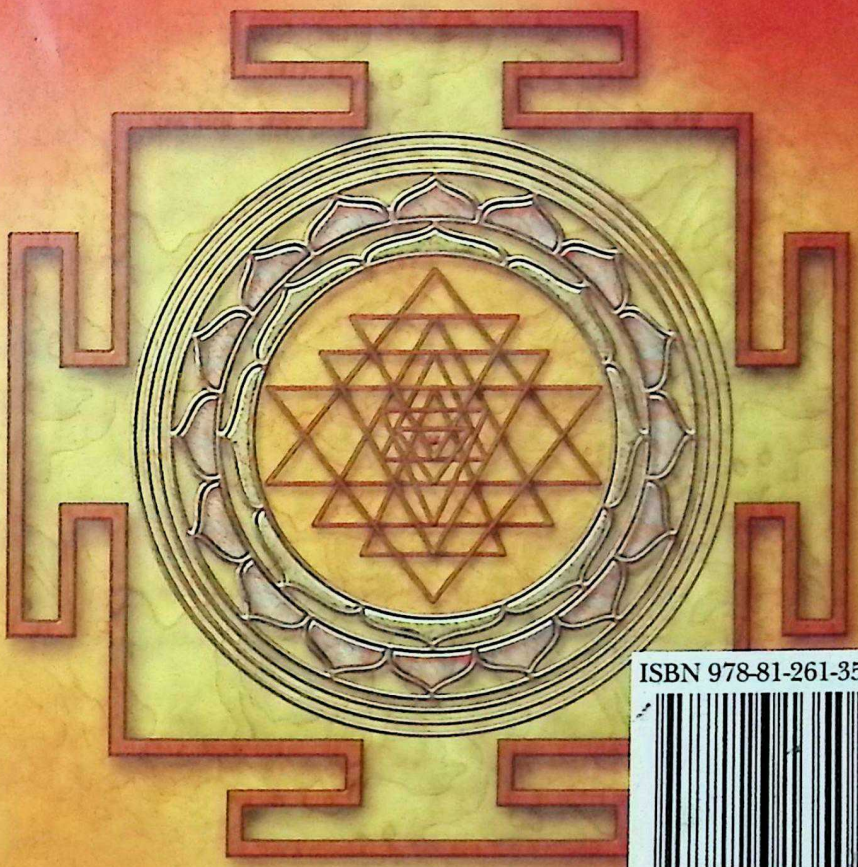

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